

# INFINITE NON-CONFORMAL ITERATED FUNCTION SYSTEMS

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**ABSTRACT.** We consider a generalisation of the self-affine iterated function systems of Lalley and Gatzouras by allowing for a countable infinity of non-conformal contractions. It is shown that the Hausdorff dimension of the limit set is equal to the supremum of the dimensions of compactly supported ergodic measures. In addition we consider the multifractal analysis for countable families of potentials. We obtain a conditional variational principle for the level sets.

## 1. INTRODUCTION

Suppose we have a compact metric space  $X$  together with a finite or countable family  $\mathcal{F} = \{S_d\}_{d \in \mathcal{D}}$  of uniformly contracting maps  $S_d : X \rightarrow X$ . The attractor  $\Lambda$  of  $\mathcal{F}$  is given by,

$$(1.1) \quad \Lambda := \bigcup_{\omega \in \mathcal{D}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} S_{\omega_1} \circ \cdots \circ S_{\omega_n}(X).$$

When  $\mathcal{F}$  consists of finitely many conformal contractions, satisfying the open set condition, the Hausdorff dimension of the attractor  $\dim_{\mathcal{H}} \Lambda$  is given by Bowen's formula as the unique zero of an associated pressure function [Bo], which is equal to the supremum over the dimensions of ergodic measures supported on the limit set (see [P] for details). When  $\mathcal{F}$  consists of a countable infinity of conformal contractions, satisfying the open set condition, the associated pressure function may not pass through zero. Nonetheless, Mauldin and Urbański have shown that  $\Lambda$  satisfies a modified version of Bowen's formula in which  $\dim_{\mathcal{H}} \Lambda$  is given by the infimum over all values for which the pressure function is negative [MU]. Moreover,  $\dim_{\mathcal{H}} \Lambda$  is equal to the supremum over the dimensions of ergodic measures supported on compact invariant subsets of the limit set.

When  $\Lambda$  is non-conformal much less is known. In [F1] Falconer showed that, when  $\mathcal{F}$  consists of finitely many affine contractions,  $\dim_{\mathcal{H}} \Lambda$  is bounded above by the unique zero of an associated subadditive pressure function. Moreover, for typical  $\mathcal{F}$ , with respect to an appropriate parameterization, this value also gives a lower bound.

However, there are very few cases in which the Hausdorff dimension of a particular non-conformal limit set is known. In most such cases the upper bound given by the subadditive pressure function is non-optimal. The first examples of this type were provided by Bedford [Be] and McMullen

[Mc]. These constructions were generalised to include a continuum of examples with variable Lyapunov exponent by Lalley and Gatzouras in [LG]. A generalisation in a different direction was given by Barański in [Bar]. In [Nu] Luzia considers certain non-conformal and non-linear repellers closely related to the self-affine limit sets of Lalley and Gatzouras. In each of these cases  $\dim_{\mathcal{H}}\Lambda$  is equal to the supremum over the dimensions of ergodic measures supported on the limit set.

Having determined the Hausdorff dimension of the the limit set in cases where the contractions are both non-conformal and have a variable Lyapunov exponent, it is natural to consider examples of iterated function systems consisting of a countable infinity of non-conformal contractions.

**Example 1** (Gauss-Rényi Products). *Given  $x, y \in [0, 1]$  and  $n \in \mathbb{N}$  we let  $a_n(x) \in \mathbb{N}$  denote the  $n$ th digit in the continued fraction expansion of  $x$  and  $b_n(y) \in \{0, 1\}$  denote the  $n$ th digit in the binary expansion of  $y$ . Choose some digit set  $\mathcal{D} \subseteq \mathbb{N} \times \{0, 1\}$  and define,*

$$\Lambda := \{(x, y) \in [0, 1]^2 : (a_n(x), b_n(y)) \in \mathcal{D} \text{ for all } n \in \mathbb{N}\}.$$

*Then  $\Lambda$  is the attractor of the iterated function system consisting of all maps of the form,*

$$(x, y) \mapsto \left( \frac{1}{a+x}, \frac{y+b}{2} \right) \text{ for } (x, y) \in [0, 1]^2,$$

*with  $(a, b) \in \mathcal{D}$ .*

This example is a member of a class of constructions which we shall refer to as INC systems (see Section 2 for the definition). For all such systems we shall show that the Hausdorff dimension of the limit system is equal to the supremum over the dimensions of ergodic measures supported on compact invariant subsets of the limit set.

We shall also consider the multifractal analysis of Birkhoff averages. When  $\mathcal{F}$  consists of finitely many conformal contractions the spectrum is well understood [BS, FFW, O, OW, PW]; the dimension of the level sets is given by a conditional variational principle (see [B, Chapter 9] for details). For a useful survey on related multifractal results we recommend [Cl].

Recently there has also been a great deal of work dealing with cases in which  $\mathcal{F}$  consists of a countable infinity of conformal contractions (see [JK, KMS, KS, IJ, FLM, FLWWJ, FLMW]). In this setting the Birkhoff spectra can display a wide variety of interesting behaviour. For example, due to the unbounded contraction rates one can have phase transitions and flat regions in the spectrum (see [KMS, IJ]). In addition, when dealing with a countable infinity of potentials on an infinite IFS, one does not obtain the usual conditional variational principle (see [FLMW, Theorem 1.1] for example). This is a consequence of the lack of both compactness and upper semi-continuity of entropy for countable state systems.

There has also been some work on the multifractal analysis of Birkhoff averages for  $\mathcal{F}$  consisting of finitely many affine planar contractions with a

diagonal linear part. In [JS] Jordan and Simon gave a conditional variational principle which holds for typical members of parameterizable families of examples. In [BM, BF] Barral, Mensi and Feng investigated the multifractal analysis of Birkhoff averages on the self-affine limit sets of Bedford and McMullen [Be, Mc]. In particular, they obtain a conditional variational principle for the level sets [BF]. In [R] this result is extended to include the self-affine limit sets of Lalley and Gatzouras. However, the method used in [R] relies heavily upon the compactness of the associated symbolic space.

In this paper we shall consider the multifractal analysis of Birkhoff averages for a family of iterated function systems consisting of a countable infinity of non-conformal contractions which we shall refer to as INC systems. We shall obtain a conditional variational principle for the level sets of a countable infinity of Birkhoff averages on the limit set for an INC system.

## 2. NOTATION AND STATEMENT OF RESULTS

Let  $I := [0, 1]$  denote the closed unit interval. Given a digit set  $\mathcal{B}$  we let  $\mathcal{B}^* := \bigcup_{n \in \mathbb{N}} \mathcal{B}^n$  denote the space of all finite strings. Given a sequence of maps  $\{f_j\}_{j \in \mathcal{B}}$  indexed by  $\mathcal{B}$  and a finite string  $\omega = (\omega_1, \dots, \omega_n) \in \mathcal{B}^*$  we let  $f_\omega$  denote the composition  $f_\omega := f_{\omega_1} \circ \dots \circ f_{\omega_n}$ .

**Definition 2.1** (Interval Iterated Function Systems). *By an interval iterated function system we shall mean a family  $\{f_j : j \in \mathcal{B}\}$  of  $C^1$  maps  $f_j : I \rightarrow I$ , indexed over some finite or countable digit set  $\mathcal{B}$ , which satisfies the following assumptions.*

(UCC) *Uniform Contraction Condition. There exists a contraction ratio  $\xi \in (0, 1)$  and  $N \in \mathbb{N}$  such that for all  $n \geq N$  and all  $\omega \in \mathcal{B}^n$  we have*

$$\sup_{x \in I} |f'_\omega(x)| \leq \xi^n.$$

(OIC) *Open Interval Condition. For all  $j_1, j_2 \in \mathcal{B}$  with  $j_1 \neq j_2$ , we have*

$$f_{j_1}((0, 1)) \cap f_{j_2}((0, 1)) = \emptyset.$$

(TDP) *Tempered Distortion Property. There exists some sequence  $\rho_n$  with  $\lim_{n \rightarrow \infty} \rho_n = 0$  such that for all  $n \in \mathbb{N}$  and for all  $\omega \in \mathcal{B}^n$  and all  $x, y \in I$  we have*

$$e^{-n\rho_n} \leq \frac{|f'_\omega(x)|}{|f'_\omega(y)|} \leq e^{n\rho_n}.$$

*If  $\mathcal{B}$  is finite then  $\{f_j : j \in \mathcal{B}\}$  is said to be a finite interval iterated function system.*

**Definition 2.2** (INC Systems). *Suppose we have a finite interval iterated function system  $\{g_i : i \in \mathcal{A}\}$  and for each  $i \in \mathcal{A}$  we have a (finite or countable) interval iterated function system  $\{f_{ij} : j \in \mathcal{B}_i\}$  with  $\sup_{x \in I} |f'_{ij}(x)| \leq$*

$\inf_{x \in \mathcal{I}} |g'_i(x)|$  for each  $j \in \mathcal{B}_i$ . Let  $\mathcal{D} := \{(i, j) : i \in \mathcal{A}, j \in \mathcal{B}_i\}$  and for each pair  $(i, j) \in \mathcal{D}$  we let  $S_{ij}$  denote the map given by

$$S_{ij}(x, y) = (f_{ij}(x), g_i(y)) \text{ for } (x, y) \in I^2.$$

An iterated function system  $\{S_{ij} : (i, j) \in \mathcal{D}\}$  defined in this way shall be referred to as an *INC System*.

We shall use the symbolic spaces  $\Sigma := \mathcal{D}^{\mathbb{N}}$ , and  $\Sigma_v := \mathcal{A}^{\mathbb{N}}$ , each of which is endowed with the product topology. Let  $\sigma : \Sigma \rightarrow \Sigma$  and  $\sigma_v : \Sigma_v \rightarrow \Sigma_v$  denote the corresponding left shift operators. We let  $\pi : \Sigma \rightarrow \Sigma_v$  denote the projection given by  $\pi(\omega) = (i_\nu)_{\nu \in \mathbb{N}}$  for  $\omega = ((i_\nu, j_\nu))_{\nu \in \mathbb{N}} \in \Sigma$ . We also let  $\pi((i_\nu, j_\nu)_{\nu=1}^n) = (i_\nu)_{\nu=1}^n$  for a finite string  $(i_\nu, j_\nu)_{\nu=1}^n \in \mathcal{D}^n$ . We define a projection  $\Pi : \Sigma \rightarrow I^2$  by

$$(2.1) \quad \Pi(\omega) := \lim_{n \rightarrow \infty} S_{\omega_1} \circ \cdots \circ S_{\omega_n} (I^2) \text{ for } \omega = (\omega_n)_{n \in \mathbb{N}} \in \Sigma.$$

We also define a vertical projection  $\Pi_v : \Sigma_v \rightarrow I$  by

$$(2.2) \quad \Pi_v(\mathbf{i}) := \lim_{n \rightarrow \infty} g_{i_1} \circ \cdots \circ g_{i_n} (I^2) \text{ for } \mathbf{i} = (i_\nu)_{\nu \in \mathbb{N}} \in \Sigma_v.$$

Let  $\Lambda := \Pi(\Sigma)$ . It follows that,

$$(2.3) \quad \Lambda = \bigcup_{(i,j) \in \mathcal{D}} S_{ij}(\Lambda).$$

Given any finite subset  $\mathcal{F} \subset \mathcal{D}$  we let  $\Lambda_{\mathcal{F}}$  denote the unique non-empty compact set satisfying,

$$(2.4) \quad \Lambda_{\mathcal{F}} = \bigcup_{(i,j) \in \mathcal{F}} S_{ij}(\Lambda).$$

In addition we define  $\chi \in \Sigma \rightarrow \mathbb{R}$  and  $\psi \in \Sigma_v \rightarrow \mathbb{R}$  by

$$(2.5) \quad \chi(\omega) := -\log |f'_{\omega_1}(\Pi(\sigma\omega))| \text{ for } \omega = (\omega_\nu)_{\nu \in \mathbb{N}} \in \Sigma,$$

$$(2.6) \quad \psi(\mathbf{i}) := -\log |g'_{i_1}(\Pi_v(\sigma\mathbf{i}))| \text{ for } \mathbf{i} = (i_\nu)_{\nu \in \mathbb{N}} \in \Sigma_v.$$

Let  $\mathcal{A}$  denote the Borel sigma algebra on  $\Sigma_v$ . We let  $\mathcal{M}_\sigma(\Sigma)$  denote the set of all  $\sigma$ -invariant Borel probability measures on  $\Sigma$  and let  $\mathcal{M}_\sigma^*(\Sigma)$  denote the set of  $\mu \in \mathcal{M}_\sigma(\Sigma)$  which are supported on a compact subset of  $\Sigma$ . Similarly we let  $\mathcal{E}_\sigma(\Sigma)$  denote the set of  $\mu \in \mathcal{M}_\sigma(\Sigma)$  which are ergodic and  $\mathcal{E}_\sigma^*(\Sigma)$  denote the set of  $\mu \in \mathcal{E}_\sigma(\Sigma)$  which are compactly supported. Given  $\mu \in \mathcal{M}_\sigma^*(\Sigma)$  we define

$$(2.7) \quad D(\mu) := \frac{h_\mu(\sigma | \pi^{-1} \mathcal{A})}{\int \chi d\mu} + \frac{h_{\mu \circ \pi^{-1}}(\sigma_v)}{\int \psi d\mu \circ \pi^{-1}}.$$

By the Ledrappier and Young dimension formula (see [LY, Corollary D])  $D(\mu)$  gives the dimension of  $\mu$  for all  $\mu \in \mathcal{E}_\sigma^*(\Sigma)$ .

**Theorem 1.** *Let  $\Lambda$  be the attractor of an INC system. Then,*

$$\begin{aligned} \dim_{\mathcal{H}} \Lambda &= \sup \{D(\mu) : \mu \in \mathcal{E}_{\sigma}^*(\Sigma)\} \\ &= \sup \{D(\mu) : \mu \in \mathcal{M}_{\sigma}^*(\Sigma)\} \\ &= \sup \{\dim_{\mathcal{H}} \Lambda_{\mathcal{F}} : \mathcal{F} \text{ is a finite subset of } \mathcal{D}\}. \end{aligned}$$

Given a potential  $\varphi : \Sigma \rightarrow \mathbb{R}$  and  $n \in \mathbb{N}$  we shall let  $S_n(\varphi) := \sum_{l=0}^{n-1} \varphi \circ \sigma^l$ ,  $A_n(\varphi) := n^{-1} S_n(\varphi)$  and define

$$\text{var}_n(\varphi) := \sup \{|\varphi(\omega) - \varphi(\tau)| : \omega, \tau \in \Sigma \text{ with } \omega_l = \tau_l \text{ for } l = 1, \dots, n\}.$$

**Definition 2.3** (Tempered Distortion Property). *A potential  $\varphi : \Sigma \rightarrow \mathbb{R}$  is said to satisfy the tempered distortion property if  $\lim_{n \rightarrow \infty} \text{var}_n(A_n(\varphi)) = 0$ .*

It follows from the tempered distortion property of  $\{f_{ij} : j \in \mathcal{B}_i\}$  and  $\{g_i : i \in \mathcal{A}\}$  (see Definition 2.1 (TDP)) that both  $\chi$  and  $\psi \circ \pi$  satisfy the tempered distortion property in Definition 2.3. We shall focus on potentials satisfying the tempered distortion property which are bounded on one side. That is, there exists some  $a \in \mathbb{R}$  such that either  $\varphi(\omega) \leq a$  for all  $\omega \in \Sigma$  or  $\varphi(\omega) \geq a$  for all  $\omega \in \Sigma$ . Note that this family includes every positive valued uniformly continuous potential.

Suppose we have a countable family  $(\varphi_k)_{k \in \mathbb{N}}$  of real-valued potentials  $\varphi_k : \Sigma \rightarrow \mathbb{R}$ , together with some  $(\alpha_k)_{k \in \mathbb{N}} \subset \mathbb{R} \cup \{-\infty, +\infty\}$ . We define,

$$(2.8) \quad E_{\varphi}(\alpha) := \left\{ \omega \in \Sigma : \lim_{n \rightarrow \infty} A_n(\varphi_k)(\omega) = \alpha_k \text{ for all } k \in \mathbb{N} \right\},$$

and let  $J_{\varphi}(\alpha) := \Pi(E_{\varphi}(\alpha))$ . Here limits are taken with respect to the usual two point compactification of  $\mathbb{R}$ .

Given  $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$  we define a shrinking family  $\{B_m(\alpha)\}_{m \in \mathbb{N}}$  of neighbourhoods of  $\alpha$  by,

$$(2.9) \quad B_m(\alpha) := \begin{cases} \{x : |x - \alpha| < \frac{1}{m}\} & \text{if } \alpha \in \mathbb{R} \\ (m, +\infty) & \text{if } \alpha = +\infty \\ (-\infty, -m) & \text{if } \alpha = -\infty. \end{cases}$$

**Theorem 2.** *Suppose we have countably many potentials  $(\varphi_k)_{k \in \mathbb{N}}$  each of which satisfies the tempered distortion property and is bounded on one side. Then, for all  $\alpha = (\alpha_k)_{k \in \mathbb{N}} \in (\mathbb{R} \cup \{-\infty, +\infty\})^{\mathbb{N}}$  we have,*

$$\begin{aligned} \dim_{\mathcal{H}} J_{\varphi}(\alpha) &= \lim_{m \rightarrow \infty} \sup \left\{ D(\mu) : \mu \in \mathcal{E}_{\sigma}^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\} \\ &= \lim_{m \rightarrow \infty} \sup \left\{ D(\mu) : \mu \in \mathcal{M}_{\sigma}^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}. \end{aligned}$$

Note that in general it is impossible to remove the dependence on  $m$  and obtain a variational principle of the form [B, Theorem 9.1.4]. This is a consequence of lack of compactness in the symbolic space, along with

the lack of upper semi-continuity for entropy. Example 4 illustrates this phenomenon.

Nonetheless for the interior of the spectrum for a single potential we can use an argument from Iommi and Jordan [IJ] to recover the usual conditional variational principle. Let  $\alpha_{\min} := \inf \left\{ \int \varphi d\mu : \mu \in \mathcal{M}_\sigma(\Sigma) \right\}$  and  $\alpha_{\max} := \sup \left\{ \int \varphi d\mu : \mu \in \mathcal{M}_\sigma(\Sigma) \right\}$ .

**Theorem 3.** *Given a non-negative potential  $\varphi : \Sigma \rightarrow \mathbb{R}$  satisfying the tempered distortion property and some  $\alpha \in (\alpha_{\min}, \alpha_{\max})$  we have*

$$\dim_{\mathcal{H}} J_\varphi(\alpha) = \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma^*(\Sigma), \int \varphi d\mu = \alpha \right\}.$$

*Proof.* One can argue as in [IJ, Lemma 3.2], by taking convex combinations, to see that the supremum on the right depends continuously on  $\alpha$ . Consequently,

$$\begin{aligned} \lim_{m \rightarrow \infty} \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma^*(\Sigma), \int \varphi d\mu \in B_m(\alpha) \right\} \\ \leq \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma^*(\Sigma), \int \varphi d\mu = \alpha \right\}. \end{aligned}$$

Thus, Theorem 3 follows from Theorem 2.  $\square$

The following examples are applications of Theorem 2.

**Example 2** (Geometric Arithmetic Mean Sets). *Let  $\Lambda$  be as in Example 1. For each  $\alpha, \beta \in \mathbb{R}$  we define,*

$$\Lambda^\times(\alpha, \beta) := \left\{ (x, y) \in \Lambda : \lim_{n \rightarrow \infty} \sqrt[n]{a_1(x) \cdots a_n(x)} = \alpha, \lim_{n \rightarrow \infty} \frac{b_1(y) + \cdots + b_n(y)}{n} = \beta \right\}.$$

*Then  $\dim_{\mathcal{H}} \Lambda^\times(\alpha, \beta)$  varies continuously as a function of  $(\alpha, \beta) \in \mathbb{R}^2$ .*

**Example 3** (Arithmetic Mean Sets). *Let  $\Lambda$  be as in Example 1. For each  $\alpha, \beta \in \mathbb{R}$  we define,*

$$\Lambda^+(\alpha, \beta) := \left\{ (x, y) \in \Lambda : \lim_{n \rightarrow \infty} \frac{a_1(x) + \cdots + a_n(x)}{n} = \alpha, \lim_{n \rightarrow \infty} \frac{b_1(y) + \cdots + b_n(y)}{n} = \beta \right\}.$$

*Then  $\dim_{\mathcal{H}} \Lambda^+(\alpha, \beta)$  varies continuously as a function of  $(\alpha, \beta) \in \mathbb{R}^2$ .*

**Example 4** (Total Escape of Mass). *Within the setting of Example 1 we consider the set,*

$$\Lambda_\infty(\mathcal{D}) := \left\{ (x, y) \in \Lambda : \lim_{n \rightarrow \infty} \frac{\#\{l \leq n : a_l(x) = m\}}{n} = 0 \text{ for all } m \in \mathbb{N} \right\}.$$

*If  $\mathcal{D}$  is finite then  $\Lambda_\infty(\mathcal{D})$  is clearly empty. However, if  $\mathcal{D} := \{(n, n \bmod 2) : n \in \mathbb{N}\}$ , then  $\dim_{\mathcal{H}} \Lambda_\infty(\mathcal{D}) = \frac{3}{2}$ .*

The rest of the paper will be directed towards proving Theorem 2, from which Theorems 3 and 1 follow. The proof will consist of an upper bound, contained in sections 3 and 4 and a lower bound, contained in sections 5 and 7. We begin the proof of the upper bound by proving an upper estimate, in Section 3, for the dimension of the level sets in the special case in which we have finitely many locally constant potentials. It is in proving this initial upper estimate that many of the difficulties lie. We use the compactness of the vertical symbolic space  $\Sigma_v = \mathcal{A}^{\mathbb{N}}$  to partition the symbolic level sets into a countable number of sets for which certain sequences depending only upon  $\pi(\omega)$  converge to some prescribed value along a sequence of good times. We then use the sequence of good times to obtain an efficient covering by approximate squares. A Misiurewicz-type argument (see [Mi]) based on [JJOP] is then used to extract a conditional  $n$ -th level Bernoulli measure for each of the horizontal fibers from the covering. Note that Misiurewicz's argument must be adjusted to deal with the lack of compactness. By weighting the horizontal fibres according to a Bernoulli measure derived from the frequencies of certain digits, along a subsequence of good times, we obtain an  $n$ -th level Bernoulli measure which not only has dimension close to the exponent given by the covering, but also integrates each of the potentials to approximately the correct value. In section 4 we apply a series of approximation arguments to deduce the upper bound given in Theorem 2 from the upper estimate from Section 3.

To prove the lower bound we use the technique of concatenating measures applied by Gelfert and Rams in [GR]. For each  $m \in \mathbb{N}$  we obtain a compactly supported ergodic measure, with near optimal dimension, which integrates each of the first  $m$  potentials to approximately the required value. By carefully concatenating a sequence of such measures it is possible to obtain a measure for which typical points, with respect to that measure, have local dimension equal to the expression in Theorem 2 and for which each of the countably many Birkhoff averages converge to the required value.

### 3. THE UPPER BOUND FOR LOCALLY CONSTANT POTENTIALS

In this section we shall make the following simplifying assumptions. Firstly, we will suppose that there exists a contraction ratio  $\zeta \in (0, 1)$  such that for each  $i \in \mathcal{A}$ ,  $\sup_{x \in I} |g'_i(x)| \leq \zeta$ . Secondly, we will suppose that we have finitely many potentials,  $\varphi^1, \dots, \varphi^K$ , each of which is both locally constant and bounded below by 1. That is, for each  $k = 1, \dots, K$ , there exists a  $\mathcal{D}$ -sequence  $(\varphi_{ij}^k)_{(i,j) \in \mathcal{D}}$  such that  $\varphi^k(\omega) = \varphi_{\omega_1}^k \geq 1$  for all  $\omega = (\omega_\nu)_{\nu \in \mathbb{N}} \in \Sigma$ .

We shall often view the  $K$ -tuple of potentials,  $\varphi^1, \dots, \varphi^K$  as a single vector valued potential  $\varphi : \omega \mapsto (\varphi_{\omega_1}^k)_{k=1}^K$ , taking values in  $\mathbb{R}^K$ . We endow  $\mathbb{R}^K$  with the supremum metric, which we shall denote by  $\|\cdot\|_\infty$ , as well as the usual partial order given by  $(c_k)_{k=1}^K \leq (d_k)_{k=1}^K$  if and only if  $c_k \leq d_k$  for all  $k = 1, \dots, K$ . We also let  $[c, d] := \{x \in \mathbb{R}^K : c \leq x \leq d\}$ .

Let  $\mathbb{R} \cup \{-\infty, +\infty\}$  denote the usual two-point compactification of  $\mathbb{R}$ . Given a sequence of real numbers  $(a_n)_{n \in \mathbb{N}}$  we let  $\Omega((a_n)_{n \in \mathbb{N}})$  denote its set of accumulation points in  $\mathbb{R} \cup \{-\infty, +\infty\}$ . For each  $k = 1, \dots, K$ , we fix some (possibly infinite) interval  $\Gamma_k = [\gamma_{\min}^k, \gamma_{\max}^k] \subseteq \mathbb{R} \cup \{+\infty\}$ , let  $\Gamma := \prod_{k=1}^K \Gamma_k = [\gamma_{\min}, \gamma_{\max}]$ , where  $\gamma_{\min} := (\gamma_{\min}^k)_{k=1}^K$  and  $\gamma_{\max} := (\gamma_{\max}^k)_{k=1}^K$ . Define,

$$(3.1) \quad E_\varphi(\Gamma) := \{\omega \in \Sigma : \Omega((A_n(\varphi)(\omega))_{n \in \mathbb{N}}) \subseteq \Gamma\},$$

and let  $J_\varphi(\Gamma) := \Pi(E_\varphi(\Gamma))$ .

For each  $(i, j) \in \mathcal{D}$  we let  $b_i := \sup_{x \in I} |g'_i(x)|$  and  $a_{ij} := \sup_{x \in I} |f'_{ij}(x)|$ . We define  $\tilde{\chi} : \Sigma \rightarrow \mathbb{R}$  and  $\tilde{\chi}^v : \Sigma_v \rightarrow \mathbb{R}$  by

$$(3.2) \quad \tilde{\chi}(\omega) := -\log a_{\omega_1} \text{ for } \omega = (\omega_\nu)_{\nu \in \mathbb{N}} \in \Sigma,$$

$$(3.3) \quad \tilde{\psi}(\mathbf{i}) := -\log b_{i_1} \text{ for } \mathbf{i} = (i_\nu)_{\nu \in \mathbb{N}} \in \Sigma_v.$$

Given  $q \in \mathbb{N}$  and  $\mu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$  we define

$$(3.4) \quad \tilde{D}_q(\mu) := \frac{h_\mu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\tilde{\chi}) d\mu} + \frac{h_{\mu \circ \pi^{-1}}(\sigma_v^q)}{\int S_q(\tilde{\psi}) d\mu \circ \pi_v^{-1}}.$$

**Proposition 3.1.**

$$\dim_{\mathcal{H}} J_\varphi(\alpha) \leq \lim_{\xi \rightarrow 0} \left\{ \tilde{D}_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \int A_q(\varphi) d\mu \in [\gamma_{\min} - \xi, \gamma_{\max} + \xi] \right\}.$$

**3.1. Building a Cover.** Define

$$(3.5) \quad L_n(\omega) := \min \left\{ l \geq 1 : \prod_{\nu=1}^l b_{i_\nu} \leq \prod_{\nu=1}^n a_{i_\nu j_\nu} \right\}.$$

Note that this implies

$$(3.6) \quad 1 \leq \frac{\prod_{\nu=1}^n a_{i_\nu j_\nu}}{\prod_{\nu=1}^{L_n(\omega)} b_{i_\nu}} < b_{\min}^{-1}.$$

Moreover, since  $a_{ij} \leq b_i$  for all  $(i, j) \in \mathcal{D}$ ,  $L_n(\omega) \geq n$ .

Given  $(\omega_\nu)_{\nu=1}^n = ((i_\nu, j_\nu))_{\nu=1}^n \in \mathcal{D}^n$  we let

$$(3.7) \quad [\omega_1 \cdots \omega_n] := \{\omega' \in \Sigma : \omega'_\nu = \omega_\nu \text{ for } \nu = 1, \dots, n\}$$

and

$$(3.8) \quad [i_1 \cdots i_n] := \{\mathbf{i}' \in \Sigma_v : i'_\nu = i_\nu \text{ for } \nu = 1, \dots, n\}.$$

Given  $\omega = ((i_\nu, j_\nu))_{\nu=1}^\infty \in \Sigma$  we let  $B_n(\omega)$  denote the  $n$ th approximate square,

$$(3.9) \quad B_n(\omega) := \Pi([\omega_1 \cdots \omega_n] \cap \sigma^{-n} \pi^{-1} [i_{n+1} \cdots i_{L_n(\omega)}]).$$

Thus,

$$(3.10) \quad \text{diam}(B_n(\omega)) \leq \max \left\{ \left( \prod_{\nu=1}^n a_{i_\nu j_\nu} \right), b_{\min}^{-1} \left( \prod_{\nu=1}^{L_n(\omega)} b_{i_\nu} \right) \right\}.$$



We also define a map  $\phi_i : \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{D}^n \rightarrow \{i\} \times \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{B}_i^n$  for each  $i \in \mathcal{A}$  by

$$\phi_i : ((i'_1, j'_1), (i'_1, j'_2), \dots, (i'_1, j'_{\nu_1})) \mapsto ((i, j'_{\nu_1}), (i, j'_{\nu_2}), \dots, (i, j'_{\nu_{n_i}})),$$

where  $\nu_1 < \nu_2 < \dots < \nu_{n_i}$  and  $\{\nu_l\}_{l=1}^{n_i} = \{r \leq n : i'_r = i\}$ .

Given  $q \in \mathbb{N}$  we define,

$$\begin{aligned} \mathbb{P}_q(\mathcal{A}) &:= \left\{ (p_i)_{i \in \mathcal{A}^q} \in [0, 1]^{\mathcal{A}^q} : \sum_{i \in \mathcal{A}^q} p_i = 1 \right\}, \\ \mathbb{Q}_q(\mathcal{A}) &:= \{ (p_i)_{i \in \mathcal{A}^q} \in \mathbb{P}_q(\mathcal{A}) : p_i \in \mathbb{Q} \setminus \{0\} \text{ for each } i \in \mathcal{A}^q \}. \end{aligned}$$

Each  $\mathbb{P}_q(\mathcal{A})$  is given the maximum norm  $\|\cdot\|_\infty$ . Note that for each  $q \in \mathbb{N}$ ,  $\mathbb{P}_q(\mathcal{A})$  is compact and  $\mathbb{Q}_q(\mathcal{A})$  is a dense countable subset. We let  $\mathbb{P}(\mathcal{A}) := \mathbb{P}_1(\mathcal{A})$  and  $\mathbb{Q}(\mathcal{A}) := \mathbb{Q}_1(\mathcal{A})$ .

Given  $\mathbf{p} = (p_i)_{i \in \mathcal{A}^q} \in \mathbb{P}_q(\mathcal{A})$  we define,

$$(3.11) \quad d_q(\mathbf{p}) := \frac{\sum_{i \in \mathcal{A}^q} p_i \log p_i}{\sum_{i \in \mathcal{A}^q} p_i \log b_i} = \frac{h_{\mu_{\mathbf{p}}}(\sigma_v^q)}{\int S_q(\tilde{\psi}) d\mu_{\mathbf{p}}},$$

where  $\mu_{\mathbf{p}}$  denotes the  $q$ -th level Bernoulli measure on  $\Sigma_v$  defined by  $\mu_{\mathbf{p}}([i]) = p_i$  for all  $i \in \mathcal{A}^q$ . We let  $d(\mathbf{p}) := d_1(\mathbf{p})$  for  $\mathbf{p} \in \mathbb{P}(\mathcal{A})$ .

Given  $\rho \in \mathbb{Q}(\mathcal{A})$ ,  $n \in \mathbb{N}$ , and  $\lambda = (\lambda^k)_{k=1}^K \in \mathbb{Q}(\mathcal{A})^K$  with  $\lambda^k = (\lambda_i^k)_{i \in \mathcal{A}}$  for each  $k = 1, \dots, K$  we define,

$$(3.12) \quad \mathcal{B}_i^{n, \epsilon}(\Gamma, \rho, \lambda) := \left\{ (ij_\nu)_{\nu=1}^l : l = \rho_i n \pm \epsilon n, \sum_{\nu=1}^l \varphi_{ij_\nu}^k \pm \epsilon n \in \lambda_i^k \Gamma \right\},$$

for each  $i \in \mathcal{A}$  and let

$$(3.13) \quad \mathcal{B}^{n, \epsilon}(\Gamma, \rho, \lambda) := \left\{ (\vartheta^i)_{i \in \mathcal{A}} \in \prod_{i \in \mathcal{A}} \mathcal{B}_i^{n, \epsilon}(\Gamma, \rho, \lambda) : \sum_{i \in \mathcal{A}} |\vartheta^i| = n \right\}.$$

Now define,

$$\begin{aligned} s_{n, \epsilon}(\Gamma, \rho, \lambda) &:= \inf \left\{ s : \sum_{(\vartheta^i)_{i \in \mathcal{A}} \in \mathcal{B}^{n, \epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta^i}^s \leq 1 \right\}, \\ s_\epsilon(\Gamma, \rho, \lambda) &:= \limsup_{n \rightarrow \infty} s_{n, \epsilon}(\Gamma, \rho, \lambda), \\ \delta_\epsilon(\Gamma) &:= \sup \{ s_\epsilon(\Gamma, \rho, \lambda) + d(\rho) : \rho \in \mathbb{Q}(\mathcal{A}), \lambda \in \mathbb{Q}(\mathcal{A})^K \}, \\ \delta(\Gamma) &:= \liminf_{\epsilon \rightarrow 0} \delta_\epsilon(\Gamma). \end{aligned}$$

**Lemma 3.1** (Building a Cover).  $\dim_{\mathcal{H}} J_\varphi(\Gamma) \leq \delta(\Gamma)$ .

*Proof.* Take some  $\xi > 0$ . Note that the map  $\mathbf{p} \mapsto d(\mathbf{p})$  defines a continuous function on the compact space  $\mathbb{P}(\mathcal{A})$ . Consequently there exists some  $\epsilon > 0$  such that  $\delta_\epsilon(\Gamma) < \delta(\Gamma) + \xi$  and for all  $\mathbf{p}, \mathbf{q} \in \mathbb{P}(\mathcal{A})$  with  $\|\mathbf{p} - \mathbf{q}\|_\infty < \epsilon$  we have  $|d(\mathbf{p}) - d(\mathbf{q})| < \xi$ .

We shall define a function  $F_\xi : \Sigma \rightarrow \mathbb{Q}(\mathcal{A})^{2+K}$  in the following way. Given  $\omega \in \Sigma$  we extract a subsequence  $(n_q)_{q \in \mathbb{N}}$  satisfying,

$$\begin{aligned} \text{(i)} \quad & \lim_{q \rightarrow \infty} \frac{\sum_{i \in \mathcal{A}} P_i(\omega|n_q) \log P_i(\omega|n_q)}{\sum_{i \in \mathcal{A}} P_i(\omega|n_q) \log b_i} = \limsup_{n \rightarrow \infty} \frac{\sum_{i \in \mathcal{A}} P_i(\omega|n) \log P_i(\omega|n)}{\sum_{i \in \mathcal{A}} P_i(\omega|n) \log b_i}, \\ \text{(ii)} \quad & \lim_{q \rightarrow \infty} (P_i(\omega|n_q))_{i \in \mathcal{A}} = P(\omega) = (P_i(\omega))_{i \in \mathcal{A}}, \\ \text{(iii)} \quad & \lim_{q \rightarrow \infty} (P_i(\omega|L_{n_q}(\omega)))_{i \in \mathcal{A}} = Q(\omega) = (Q_i(\omega))_{i \in \mathcal{A}}, \\ \text{(iv)} \quad & \lim_{q \rightarrow \infty} \left( \frac{\sum_{j \in \mathcal{B}_i} P_{ij}(\omega|n_q) \varphi_{ij}^k}{\sum_{(i', j') \in \mathcal{D}} P_{i'j'}(\omega|n_q) \varphi_{i'j'}^k} \right)_{i \in \mathcal{A}} = R^k(\omega) = (R_i^k(\omega))_{i \in \mathcal{A}}, \end{aligned}$$

for each  $k = 1, \dots, K$ . We let  $R(\omega) := (R^k(\omega))_{k=1}^K$ . Note that by (i) we always have  $d(Q(\omega)) \leq d(P(\omega))$ . Since  $\mathbb{Q}(\mathcal{A})$  is dense in  $\mathbb{P}(\mathcal{A})$  we may choose  $\kappa(\omega) = (\kappa_i(\omega))_{i \in \mathcal{A}} \in \mathbb{Q}(\mathcal{A})$  so that  $\kappa_i(\omega) > P_i(\omega)\zeta^\xi$  for each  $i \in \mathcal{A}$ . We choose  $\rho(\omega) \in \mathbb{Q}(\mathcal{A})$  and  $\lambda(\omega) \in \mathbb{Q}(\mathcal{A})^K$  so that  $\|P(\omega) - \rho(\omega)\| < \epsilon$  and  $\|R(\omega) - \lambda(\omega)\|_\infty < \epsilon$ . Let  $F_\xi(\omega) := (\rho(\omega), \kappa(\omega), \lambda(\omega))$ .

Define,

$$\begin{aligned} E_\varphi^{(\rho, \kappa, \lambda)}(\Gamma) &:= E_\varphi(\Gamma) \cap F_\xi^{-1}(\rho, \kappa, \lambda), \\ J_\varphi^{(\rho, \kappa, \lambda)}(\Gamma) &:= \Pi(E_\varphi^{(\rho, \kappa, \lambda)}(\Gamma)). \end{aligned}$$

Since  $\mathbb{Q}(\mathcal{A})^{2+K}$  is countable, to show that

$$\dim_{\mathcal{H}} J_\varphi(\Gamma) \leq \delta(\Gamma) + 6\xi,$$

it suffices to fix  $(\rho, \kappa, \lambda) \in \mathbb{Q}(\mathcal{A})^{2+K}$  and show that

$$\dim_{\mathcal{H}} J_\varphi^{(\rho, \kappa, \lambda)}(\Gamma) \leq \delta(\Gamma) + 6\xi.$$

By the definition of  $s_\epsilon(\Gamma, \rho, \lambda)$  we may take  $N(\epsilon) \in \mathbb{N}$  so that for all  $n \geq N(\epsilon)$  we have

$$(3.14) \quad \sum_{(\vartheta^i)_{i \in \mathcal{A}} \in \mathcal{B}^{n, \epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta^i}^{s_\epsilon(\Gamma, \rho, \lambda) + \xi} < 1,$$

and hence,

$$(3.15) \quad \sum_{(\vartheta^i)_{i \in \mathcal{A}} \in \mathcal{B}^{n, \epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta^i}^{s_\epsilon(\Gamma, \rho, \lambda) + 2\xi} < \zeta^{n\xi}.$$

Given  $\omega \in E_\varphi^{(\rho, \kappa, \lambda)}(\Gamma)$  we have,

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{\sum_{i \in \mathcal{A}} P_i(\omega|L_{n_q}(\omega)) \log \rho_i}{\sum_{i \in \mathcal{A}} P_i(\omega|L_{n_q}(\omega)) \log b_i} &\leq \lim_{q \rightarrow \infty} \frac{\sum_{i \in \mathcal{A}} P_i(\omega|L_{n_q}(\omega)) \log P_i(\omega|L_{n_q}(\omega))}{\sum_{i \in \mathcal{A}} P_i(\omega|L_{n_q}(\omega)) \log b_i} + \xi \\ &\leq d(Q(\omega)) + \xi \leq d(P(\omega)) + \xi \\ &< d(\rho) + 2\xi. \end{aligned}$$

Thus, for all sufficiently large  $q$  we have,

$$\begin{aligned} \text{diam}(B_{n_q}(\omega))^{d(\rho)+3\xi} &\leq b_{\min}^{-d(\rho)-2\xi} \left( b_{i_1} \cdots b_{i_{L_{n_q}(\omega)}} \right)^{d(\rho)+2\xi} \\ &\leq b_{\min}^{-d(\rho)-2\xi} \left( \rho_{i_1} \cdots \rho_{i_{L_{n_q}(\omega)}} \right) \zeta^{L_{n_q}(\omega)\xi}. \end{aligned}$$

We also have,

$$\text{diam}(B_{n_q}(\omega)) \leq \prod_{\nu=1}^{n_q} a_{i_\nu j_\nu} \leq \prod_{i \in \mathcal{A}} a_{\phi^i(\omega|n_q)}.$$

Moreover, by (2) and (4) we also have  $\phi^i(\omega|n_q) \in \mathcal{B}_i^{n_q, \epsilon}(\Gamma, \rho, \lambda)$  for each  $i \in \mathcal{A}$  and hence  $(\phi^i(\omega|n_q))_{i \in \mathcal{A}} \in \mathcal{B}^{n_q, \epsilon}(\Gamma, \rho, \lambda)$  for all sufficiently large  $q$ .

Thus, if we fix some  $r > 0$ , then for each  $\omega \in E_\varphi^{(\rho, \kappa, \lambda)}(\Gamma)$  we may take some  $n(\omega) \geq N(\epsilon)$  so that,

- (i)  $\Pi(\omega) \in B_{n(\omega)}(\omega)$ ,
- (ii)  $\text{diam}(B_{n(\omega)}(\omega)) \leq \gamma$ ,
- (iii)  $\text{diam}(B_{n(\omega)}(\omega))^{d(\rho)+3\xi} \leq b_{\min}^{-d(\rho)-2\xi} \left( \rho_{i_1} \cdots \rho_{i_{L_{n(\omega)}(\omega)}} \right) \zeta^{L_{n(\omega)}(\omega)\xi}$ ,
- (iv)  $\text{diam}(B_{n_q}(\omega))^{s_\epsilon(\Gamma, \rho, \lambda)+2\xi} \leq \prod_{i \in \mathcal{A}} a_{\phi^i(\omega|n_q)}^{s_\epsilon(\Gamma, \rho, \lambda)+2\xi}$ ,
- (v)  $(\phi^i(\omega|n(\omega)))_{i \in \mathcal{A}} \in \mathcal{B}^{n(\omega), \epsilon}(\Gamma, \rho, \lambda)$ .

Let  $\mathcal{B}_r := \left\{ B_{n(\omega)}(\omega) : \omega \in E_\varphi^{(\rho, \kappa, \lambda)}(\Gamma) \right\}$ . By (i) and (ii) above,  $\mathcal{B}_r$  forms a countable  $r$ -cover of  $J_\varphi^{(\rho, \kappa, \lambda)}(\Gamma)$ .

Note also that given  $\omega^1 = ((i_\nu^1, j_\nu^1))_{\nu \in \mathbb{N}}, \omega^2 = ((i_\nu^2, j_\nu^2))_{\nu \in \mathbb{N}} \in E_\varphi^{(\rho, \kappa, \lambda)}(\Gamma)$  with  $(i_1^1, \dots, i_{L_{n(\omega^1)}(\omega^1)}^1) = (i_1^2, \dots, i_{L_{n(\omega^2)}(\omega^2)}^2)$  and  $(\phi^i(\omega^1|n(\omega^1)))_{i \in \mathcal{A}} = (\phi^i(\omega^2|n(\omega^2)))_{i \in \mathcal{A}}$  we must have  $B_{n(\omega^1)}(\omega^1) = B_{n(\omega^2)}(\omega^2)$ . Hence,

$$\begin{aligned} &\sum_{B \in \mathcal{B}_\gamma} \text{diam}(B)^{s_\epsilon(\Gamma, \rho, \lambda)+d(\rho)+5\xi}, \\ &\leq b_{\min}^{-d(\rho)-2\xi} \sum_{l \in \mathbb{N}} \zeta^{l\xi} \left( \sum_{(i_1, \dots, i_l) \in \mathcal{A}^l} \rho_{i_1} \cdots \rho_{i_l} \right) \times \left( \sum_{n \geq N(\epsilon)} \sum_{(\vartheta^i)_{i \in \mathcal{A}} \in \mathcal{B}^{n, \epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta^i}^{s_\epsilon(\Gamma, \rho, \lambda)+2\xi} \right), \\ &\leq b_{\min}^{-d(\rho)-2\xi} \sum_{l \in \mathbb{N}} \zeta^{l\xi} \times \sum_{n \geq N(\epsilon)} \zeta^{n\xi} < \infty. \end{aligned}$$

Letting  $\gamma \rightarrow 0$  we have that

$$\begin{aligned} \dim_{\mathcal{H}} J_\varphi^{(\rho, \kappa, \lambda)}(\Gamma) &\leq s_\epsilon(\Gamma, \rho, \lambda) + d(\rho) + 5\xi, \\ &\leq \delta_\epsilon(\Gamma) + 5\xi, \\ &\leq \delta(\Gamma) + 6\xi, \end{aligned}$$

by our choice of  $\epsilon$ . Since  $\mathbb{Q}(\mathcal{A})^{2+K}$  is countable, it follows that

$$\dim_{\mathcal{H}} J_{\varphi}(\Gamma) \leq \delta(\Gamma) + 6\xi.$$

Letting  $\xi \rightarrow 0$  proves the lemma.  $\square$

**3.2. Constructing a Measure.** Define  $\mathcal{A}^{n,\epsilon}(\Gamma, \rho) \subseteq \mathcal{A}^{\lceil(1+2\epsilon)n\rceil}$  by,

$$\mathcal{A}^{n,\epsilon}(\Gamma, \rho) := \left\{ \tau \in \mathcal{A}^{\lceil(1+2\epsilon)n\rceil} : N_i(\tau) \geq (1+\epsilon)\rho_i n \text{ for each } i \in \mathcal{A} \right\}.$$

**Lemma 3.2.** *Given  $\rho \in \mathbb{P}(\mathcal{A})$  there exists  $M(\rho, \epsilon) \in \mathbb{N}$  such that for all  $n \geq M(\rho, \epsilon)$  we have,  $P^{n,\epsilon}(\Gamma, \rho) := \sum_{\tau \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho)} \rho_{\tau} > 1 - \epsilon$ .*

*Proof.* Apply Kolmogorov's strong law of large numbers and then Egorov's theorem.  $\square$

**Lemma 3.3** (Constructing a Measure).

$$\delta(\Gamma) \leq \lim_{\xi \rightarrow 0} \left\{ \tilde{D}_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma_q}^*(\Sigma), \int A_q(\varphi) d\mu \in [\gamma_{\min} - \xi, \gamma_{\max} + \xi] \right\}.$$

*Proof.* We begin by fixing some  $j_*^i \in \mathcal{B}_i$  for each  $i \in \mathcal{A}$ . We then let  $a_* := \min \{a_{ij_*^i} : i \in \mathcal{A}\}$  and  $\varphi_* := \max \{\varphi_{ij_*^i}^k : i \in \mathcal{A}, k \leq K\}$ . In what follows we shall let  $o(\epsilon)$  denote any quantity which depends only upon the observable  $\varphi$ , the iterated function system, our choice of  $(j_*^i)_{i \in \mathcal{A}}$  and  $\epsilon$ , which tends to zero as  $\epsilon$  tends to zero. Of course, the precise value of  $o(\epsilon)$  will vary from line to line.

Take  $\xi > 0$ . Then there exists  $\epsilon_0(\xi)$  such that for all  $\epsilon \leq \epsilon_0(\xi) < 1/2$  we have  $\delta_{\epsilon}(\Gamma) > \delta(\Gamma) - \xi$ . Take  $\epsilon \leq \epsilon_0(\xi)$ . Then there exists  $\rho \in \mathbb{Q}(\mathcal{A})$   $\lambda \in \mathbb{Q}(\mathcal{A})^K$  with  $s(\Gamma, \rho, \lambda) + d(\rho) > \delta(\Gamma) - \xi$ .

Consequently, there exists infinitely many  $n \in \mathbb{N}$  for which

$$(3.16) \quad \sum_{(\vartheta^i)_{i \in \mathcal{A}} \in \mathcal{B}^{n,\epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta^i}^{\delta(\Gamma) - d(\rho) - \xi} > 1.$$

In particular we may apply Lemma 3.2 and take some such  $n \geq M(\rho, \epsilon)$ , so that  $P^{n,\epsilon}(\Gamma, \rho) > 1 - \epsilon$ . By (3.16) there exists a finite subset  $\mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda) \subseteq \mathcal{B}^{n,\epsilon}(\Gamma, \rho, \lambda)$  and  $s > \delta(\Gamma) - d(\rho) - \xi$  for which

$$(3.17) \quad \sum_{(\vartheta^i)_{i \in \mathcal{A}} \in \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta^i}^s = 1.$$

Recall that we defined  $\mathcal{A}^{n,\epsilon}(\Gamma, \rho) \subseteq \mathcal{A}^{\lceil(1+2\epsilon)n\rceil}$  by,

$$\mathcal{A}^{n,\epsilon}(\Gamma, \rho) := \left\{ \tau \in \mathcal{A}^{\lceil(1+2\epsilon)n\rceil} : N_i(\tau) \geq (1+\epsilon)\rho_i n \text{ for each } i \in \mathcal{A} \right\}.$$

We now define an injective map  $\eta : \mathcal{A}^{n,\epsilon}(\Gamma, \rho) \times \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda) \rightarrow \mathcal{D}^{\lceil(1+2\epsilon)n\rceil}$  so that for all  $(\tau, (\vartheta^i)_{i \in \mathcal{A}}) \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho) \times \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda)$  and  $i \in \mathcal{A}$  we have  $\pi(\eta(\tau, (\vartheta^i)_{i \in \mathcal{A}})) = \tau$  and  $\vartheta^i$  is an initial segment of  $\phi^i(\eta(\tau, (\vartheta^i)_{i \in \mathcal{A}}))$ . To define  $\eta$  we proceed as follows. Take  $(\tau, (\vartheta^i)_{i \in \mathcal{A}}) \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho) \times \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda)$ . For each  $i \in \mathcal{A}$  we write  $\vartheta^i = ((i, j_1^i), \dots, (i, j_{m_i}^i))$  and let  $\tau = (i_1, \dots, i_{\lceil(1+2\epsilon)n\rceil})$ .

Now, for each  $\nu \in \{1, \dots, \lceil (1+2\epsilon)n \rceil\}$  we choose  $i \in \mathcal{A}$  so that  $i = i_\nu$ , and choose  $r$  so that  $\nu$  is the  $r$ -th occurrence of the digit  $i$  in  $\tau$ . If  $r \leq m_i$  then let  $\eta_\nu := (i, j_r^i)$  and if  $\nu > r$  let  $\eta_\nu = (i, j_*^i)$ . Write  $\eta(\tau, (\vartheta_i)_{i \in \mathcal{A}}) = (\eta_\nu)_{\nu=1}^n$ . Note that

$$\begin{aligned} \prod_{i \in \mathcal{A}} a_{\vartheta_i} &\geq a_{\eta(\tau, (\vartheta_i)_{i \in \mathcal{A}})} \\ &\geq \prod_{i \in \mathcal{A}} \left( a_{\vartheta_i} \times a_{ij_*^i}^{3n\epsilon} \right) \geq \left( \prod_{i \in \mathcal{A}} a_{\vartheta_i} \right) \times a_*^{3\#\mathcal{A}n\epsilon}. \end{aligned}$$

That is,

$$(3.18) \quad \log a_{\eta(\tau, (\vartheta_i)_{i \in \mathcal{A}})} = \log \left( \prod_{i \in \mathcal{A}} a_{\vartheta_i} \right) + no(\epsilon).$$

Similarly, for each  $i \in \mathcal{A}$  and  $k \in \{1, \dots, K\}$  we have

$$\begin{aligned} \rho_i(1-\epsilon)\gamma_{\min}^k n &\leq \sum_{j \in \mathcal{B}_i} N_{ij}(\vartheta_i) \varphi_{ij}^k, \\ &\leq \sum_{j \in \mathcal{B}_i} N_{ij}(\eta(\tau, (\vartheta_i)_{i \in \mathcal{A}})) \varphi_{ij}^k, \\ &\leq \sum_{j \in \mathcal{B}_i} N_{ij}(\vartheta_i) \varphi_{ij}^k + 3\epsilon n \varphi_{ij_*^i}^k, \\ &\leq \rho_i(1+\epsilon)\gamma_{\max} n + 3\epsilon n \varphi_*. \end{aligned}$$

Hence,

$$(3.19) \quad \sum_{(i,j) \in \mathcal{D}} P_{ij}(\eta(\tau, (\vartheta_i)_{i \in \mathcal{A}})) \varphi_{ij}^k \in [\gamma_{\min}^k - o(\epsilon), \gamma_{\max}^k + o(\epsilon)],$$

for each  $k = 1, \dots, K$ , and so,

$$(3.20) \quad \sum_{(i,j) \in \mathcal{D}} P_{ij}(\eta(\tau, (\vartheta_i)_{i \in \mathcal{A}})) \varphi_{ij} \in [\gamma_{\min} - o(\epsilon), \gamma_{\max} + o(\epsilon)].$$

We define a compactly supported  $\lceil (1+2\epsilon)n \rceil$ -level Bernoulli measure  $\nu$  on  $\Sigma$  in the following way. First let  $\nu(\pi^{-1}[\tau]) := \rho_\tau$  for each  $\tau \in \mathcal{A}^{\lceil (1+2\epsilon)n \rceil}$ . Then, given  $\tau \in \mathcal{A}^{\lceil (1+2\epsilon)n \rceil}$  and  $\kappa \in \mathcal{A}^{\lceil (1+2\epsilon)n \rceil}$  with  $\pi(\kappa) = \tau$  either  $\tau \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho)$  in which case we let

$$\frac{\nu([\kappa])}{\nu(\pi^{-1}[\tau])} := \begin{cases} \prod_{i \in \mathcal{A}} a_{\vartheta_i}^s & \text{if } \kappa = \eta(\tau, (\vartheta_i)_{i \in \mathcal{A}}) \text{ for some } (\vartheta_i)_{i \in \mathcal{A}} \in \prod_{i \in \mathcal{A}} \mathcal{F}_i^{n,\epsilon}(\Gamma, \rho, \lambda), \\ 0 & \text{if otherwise,} \end{cases}$$

or  $\tau = (i_\nu)_{\nu=1}^{\lceil (1+2\epsilon)n \rceil} \in \mathcal{A}^{\lceil (1+2\epsilon)n \rceil} \setminus \mathcal{A}^{n,\epsilon}(\Gamma, \rho)$ , in which case we let

$$\frac{\nu([\kappa])}{\nu(\pi^{-1}[\tau])} := \begin{cases} 1 & \text{if } \kappa = ((i_\nu, j_*^{i_\nu}))_{\nu=1}^{\lceil (1+2\epsilon)n \rceil}, \\ 0 & \text{if otherwise.} \end{cases}$$

Since  $\nu \circ \pi$  is the  $(\rho_i)_{i \in \mathcal{A}}$  Bernoulli measure on  $\mathcal{A}^{\mathbb{N}}$  we have

$$\begin{aligned} \frac{h_{\nu \circ \pi_v^{-1}}(\sigma_v^{[(1+2\epsilon)n]})}{\int S_{[(1+2\epsilon)n]}(\tilde{\psi}) d\nu \circ \pi^{-1}} &= \frac{\sum_{\tau \in \mathcal{A}^{[(1+2\epsilon)n]}} \rho_\tau \log \rho_\tau}{\sum_{\tau \in \mathcal{A}^{[(1+2\epsilon)n]}} \rho_\tau \log b_\tau} \\ &= \frac{\sum_{i \in \mathcal{A}} \rho_i \log \rho_i}{\sum_{i \in \mathcal{A}} \rho_i \log b_i} = d(\rho). \end{aligned}$$

Now define

$$\mathcal{Z}^{n,\epsilon}(\Gamma, \rho, \lambda) := \sum_{(\vartheta_i)_{i \in \mathcal{A}} \in \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta_i}^s \log \left( \prod_{i \in \mathcal{A}} a_{\vartheta_i} \right),$$

By (3.17) together with the fact that  $\log a_{ij} \leq \log \zeta < 0$  for all  $(i, j) \in \mathcal{D}$  we have,

$$\mathcal{Z}^{n,\epsilon}(\Gamma, \rho, \lambda) \leq n \log \zeta.$$

By the definition of  $\nu$  we have,

$$\begin{aligned} h_\nu(\sigma^{[(1+2\epsilon)n]} | \pi^{-1} \mathcal{A}) &= \left( \sum_{\tau \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho)} \rho_\tau \right) \sum_{(\vartheta_i)_{i \in \mathcal{A}} \in \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta_i}^s \log \left( \prod_{i \in \mathcal{A}} a_{\vartheta_i} \right) \\ &= P^{n,\epsilon}(\Gamma, \rho) \times s \mathcal{Z}_i^{n,\epsilon}(\Gamma, \rho, \lambda). \end{aligned}$$

Also, by (3.18) we have

$$\begin{aligned} \int S_{[(1+2\epsilon)n]}(\tilde{\chi}) d\nu &= \left( \sum_{\tau \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho)} \rho_\tau \right) \left( \sum_{(\vartheta_i)_{i \in \mathcal{A}} \in \mathcal{F}^{n,\epsilon}(\Gamma, \rho, \lambda)} \prod_{i \in \mathcal{A}} a_{\vartheta_i}^s \log \left( \prod_{i \in \mathcal{A}} a_{\vartheta_i} \right) - no(\epsilon) \right) \\ &\quad + \left( 1 - \sum_{\tau \in \mathcal{A}^{n,\epsilon}(\Gamma, \rho)} \rho_\tau \right) [(1+2\epsilon)n] \log a_* \\ &= P^{n,\epsilon}(\Gamma, \rho) (\mathcal{Z}_i^{n,\epsilon}(\Gamma, \rho, \lambda) - no(\epsilon)) + (1 - P^{n,\epsilon}(\Gamma, \rho)) [(1+2\epsilon)n] \log a_*. \end{aligned}$$

Since  $n \geq M(\rho, \epsilon)$  we have  $P^{n,\epsilon}(\Gamma, \rho) > 1 - \epsilon$  and consequently,

$$\frac{h_\nu(\sigma^{[(1+2\epsilon)n]} | \pi^{-1} \mathcal{A})}{\int S_{[(1+2\epsilon)n]}(\tilde{\chi}) d\nu} \geq \frac{s}{1 - o(\epsilon)}.$$

Combining this with the fact that  $s + d(\rho) > \delta(\Gamma) - \xi$  we have,

$$\begin{aligned} (3.21) \quad \tilde{D}_{[(1+2\epsilon)n]}(\nu) &\geq \frac{s}{1 - o(\epsilon)} + d(\rho) \\ &\geq \frac{s + d(\rho)}{1 - o(\epsilon)} \\ &> \frac{\delta(\Gamma) - \xi}{1 - o(\epsilon)}. \end{aligned}$$

Moreover, by the construction of  $\nu$  combined with (3.20) we have,

$$(3.22) \quad \begin{aligned} \int A_{\lceil(1+2\epsilon)n\rceil}(\varphi)d\nu &\geq P^{n,\epsilon}(\Gamma, \rho)(\gamma_{\min} - o(\epsilon)) \\ &\geq (1 - \epsilon)(\gamma_{\min} - o(\epsilon)). \end{aligned}$$

Similarly,

$$(3.23) \quad \begin{aligned} \int A_{\lceil(1+2\epsilon)n\rceil}(\varphi)d\nu &\leq P^{n,\epsilon}(\Gamma, \rho)(\gamma_{\max} + o(\epsilon)) \\ &\quad + (1 - P^{n,\epsilon}(\Gamma, \rho))\varphi_* \\ &\leq \gamma_{\max} + o(\epsilon). \end{aligned}$$

Since we can obtain such a measure  $\mu$  for all  $\epsilon \leq \epsilon_0(\xi)$ , the lemma follows by taking  $\epsilon$  sufficiently small.  $\square$

#### 4. APPROXIMATION ARGUMENTS

In this section we apply Proposition 3.1 to obtain upper estimates of increasing generality until we obtain the upper bound in Theorem 2.

We begin by dropping the assumption that our potentials  $\varphi$  are locally constant. Instead we assume that we have finitely many potentials  $\varphi_1, \dots, \varphi_K$ , with finite first level variation  $\text{var}_1(\varphi_k) < \infty$ , for each  $k = 1, \dots, K$ . We retain the assumption that for some  $\zeta \in (0, 1)$  we have  $\sup_{x \in I} |g'_i(x)| \leq \zeta$  for each  $i \in \mathcal{A}$  and also assume that  $\text{var}_1(\chi), \text{var}_1(\psi) < -\log \zeta$ . We define

$$C_\sigma(\chi, \psi) := \max \left\{ \frac{-\log \zeta}{-\log \zeta - \text{var}_1(\chi)}, \frac{-\log \zeta}{-\log \zeta - \text{var}_1(\psi)} \right\}.$$

Proposition 3.1 gives the following estimate.

**Lemma 4.1.** *Suppose we have finitely many potentials  $\varphi_1, \dots, \varphi_M$ , with  $\text{var}_1(\varphi_k) < \infty$ . Suppose also that for some  $\zeta \in (0, 1)$  we have  $\sup_{x \in I} |g'_i(x)| \leq \zeta$  for each  $i \in \mathcal{A}$  and that  $\text{var}_1(\chi), \text{var}_1(\psi) < -\log \zeta$ . Suppose that  $\alpha = (\alpha_k)_{k=1}^M$  is such that for all  $k \leq K \leq M$  we have  $\alpha_k \in \mathbb{R}$  and for  $K < k \leq M$ ,  $\alpha_k = \infty$ . Then given any  $m \in \mathbb{N}$ ,*

$$\dim_{\mathcal{H}} J_\varphi(\alpha) \leq C_\sigma(\chi, \psi) \sup \{D_q(\mu)\},$$

where the supremum is taken over all  $\mu \in \mathcal{E}_{\sigma_q}^*(\Sigma)$  for some  $q \in \mathbb{N}$  with  $|\int A_q(\varphi_k)d\mu - \alpha_k| < 3\text{var}_1(\varphi_k)$  for  $k \leq K$  and  $\int A_q(\varphi_k)d\mu > m$  for  $K < k \leq M$ .

*Proof.* For each  $k = 1, \dots, K$  we define a locally constant potential  $\tilde{\varphi}^k$  by

$$(4.1) \quad \tilde{\varphi}_k(\omega) := \sup \{ \varphi_k(\omega') : \omega_1 = \omega'_1 \},$$

for all  $\omega = (\omega_\nu)_{\nu \in \mathbb{N}} \in \Sigma$ . It follows that  $\|\varphi_k - \tilde{\varphi}_k\|_\infty < \text{var}_1(\varphi_k)$ . Thus, for all  $\omega \in E_\varphi(\alpha)$  we have  $\Omega(A_n(\tilde{\varphi}_k)) \subseteq [\alpha_k - \text{var}_1(\varphi_k), \alpha_k + \text{var}_1(\varphi_k)]$  for  $k \leq K$  and  $\Omega(A_n(\tilde{\varphi}_k)) = \{\infty\}$  for  $K < k \leq M$ , since  $\lim_{n \rightarrow \infty} A_n(\varphi_k)(\omega) = \alpha_k$  for

all  $k \leq M$ . Hence,  $J_\varphi(\alpha) \subseteq J_{\tilde{\varphi}}(\Gamma)$  where  $\Gamma := \prod_{k=1}^K [\alpha_k - \text{var}_1(\varphi_k), \alpha_k + \text{var}_1(\varphi_k)] \times \prod_{k=K+1}^M [m + 2\text{var}_1(\varphi_k), \infty]$ . Thus, by Proposition 3.1 we have,

$$\begin{aligned}
\dim_{\mathcal{H}} J_\varphi(\alpha) &\leq \dim_{\mathcal{H}} J_{\tilde{\varphi}}(\Gamma) \\
&\leq \limsup_{\xi \rightarrow 0} \left\{ \tilde{D}_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \left| \int A_q(\tilde{\varphi}_k) d\mu - \alpha_k \right| < \text{var}_1(\varphi_k) + \xi, \right. \\
&\quad \left. \text{for } k \leq K \text{ and } \int A_q(\tilde{\varphi}_k) d\mu > m + 2\text{var}_1(\varphi_k) - \xi \text{ for } K < k \leq M \right\} \\
&\leq \sup \left\{ \tilde{D}_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \left| \int A_q(\tilde{\varphi}_k) d\mu - \alpha_k \right| < 2\text{var}_1(\varphi_k), \right. \\
&\quad \left. \text{for } k \leq K \text{ and } \int A_q(\tilde{\varphi}_k) d\mu > m + \text{var}_1(\varphi_k) \text{ for } K < k \leq M \right\} \\
&\leq \sup \left\{ \tilde{D}_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \left| \int A_q(\varphi_k) d\mu - \alpha_k \right| < 3\text{var}_1(\varphi_k), \right. \\
&\quad \left. \text{for } k \leq K \text{ and } \int A_q(\varphi_k) d\mu > m \text{ for } K < k \leq M \right\}.
\end{aligned}$$

It is clear from the definitions of  $\tilde{\chi} : \Sigma \rightarrow \mathbb{R}$  and  $\zeta$  that  $\tilde{\chi}(\omega) \geq \chi(\omega) - \text{var}_1(\chi)(\omega)$  and  $\chi(\omega) \geq -\log \zeta$  for all  $\omega \in \Sigma$ . Thus  $\int S_q(\tilde{\chi}) d\mu \geq \int S_q(\chi) d\mu - q\text{var}_1(\chi) > 0$  for each  $\mu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$ , since  $\text{var}_1(\chi) < -\log \zeta$ . It follows that for each  $\mu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$  we have

$$\frac{\int S_q(\chi) d\mu}{\int S_q(\tilde{\chi}) d\mu} \leq \frac{\int S_q(\chi) d\mu}{\int S_q(\chi) d\mu - q\text{var}_1(\chi)} \leq \frac{-\log \zeta}{-\log \zeta - \text{var}_1(\chi)} \leq C_\sigma(\chi, \psi).$$

Similarly for each  $\mu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$  we have

$$\frac{\int S_q(\psi) d\mu \circ \pi^{-1}}{\int S_q(\tilde{\psi}) d\mu \circ \pi^{-1}} \leq \frac{-\log \zeta}{-\log \zeta - \text{var}_1(\psi)} \leq C_\sigma(\chi, \psi).$$

Recall that for each  $\mu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$  we defined,

$$(4.2) \quad D_q(\mu) := \frac{h_\mu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\chi) d\mu} + \frac{h_{\mu \circ \pi^{-1}}(\sigma_v^q)}{\int S_q(\psi) d\mu \circ \pi^{-1}}$$

$$(4.3) \quad \tilde{D}_q(\mu) := \frac{h_\mu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\tilde{\chi}) d\mu} + \frac{h_{\mu \circ \pi^{-1}}(\sigma_v^q)}{\int S_q(\tilde{\psi}) d\mu \circ \pi^{-1}}.$$

Thus, for each  $\mu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$  we have  $\tilde{D}_q(\mu) \leq C_\sigma(\chi, \psi) D_q(\mu)$ . The lemma follows.  $\square$

We now use the observation that an iterated N-system is itself an N-system to obtain a more refined estimate which applies in a more general situation. First recall that by the Uniform Contraction Condition, for each N system, there exists a contraction ratio  $\zeta \in (0, 1)$  and  $N \in \mathbb{N}$  such that for all  $n \geq N$  and all  $\omega \in \mathcal{D}^n$  and all  $\mathbf{i} \in \mathcal{A}^n$  we have

$$\max \left\{ \sup_{x \in I} |f'_\omega(x)|, \sup_{x \in I} |g'_\mathbf{i}(x)| \right\} \leq \zeta^n.$$



For each  $n \geq N$  we let

$$C_\sigma^n(\chi, \psi) := \max \left\{ \frac{-\log \zeta}{-\log \zeta - \text{var}_n(A_n(\chi))}, \frac{-\log \zeta}{-\log \zeta - \text{var}_n(A_n(\psi))} \right\}.$$

**Lemma 4.2.** *Suppose we have finitely many potentials  $\varphi_1, \dots, \varphi_M$ , with  $\text{var}_1(\varphi_k) < \infty$ . Suppose that  $\alpha = (\alpha_k)_{k=1}^M$  is such that for all  $k \leq K \leq M$  we have  $\alpha_k \in \mathbb{R}$  and for  $K < k \leq M$ ,  $\alpha_k = \infty$ . Fix some  $m \in \mathbb{N}$ . Then for all sufficiently large  $n \in \mathbb{N}$  we have,*

$$\dim_{\mathcal{H}} J_\varphi(\alpha) \leq C_\sigma^n(\chi, \psi) \sup \{D_q(\mu)\},$$

where the supremum is taken over all  $\mu \in \mathcal{E}_{\sigma^q}^*(\Sigma)$  for some  $q \in \mathbb{N}$  with  $|\int A_q(\varphi_k) d\mu - \alpha_k| < 3\text{var}_n(A_n(\varphi_k))$  for  $k \leq K$  and  $\int A_q(\varphi_k) d\mu > m$  for  $K < k \leq M$ .

*Proof.* First note that by the Uniform Contraction Condition, together with the Tempered Distortion Condition applied to  $\chi, \psi$  and  $\varphi_1, \dots, \varphi_K$  we may choose  $N \in \mathbb{N}$  so that for all  $n \geq N$  we have

- (i)  $\max \{\sup_{x \in I} |f'_\omega(x)|, \sup_{x \in I} |g'_1(x)|\} \leq \zeta^n$ ,
- (ii)  $\max \{\text{var}_n(A_n(\chi)), \text{var}_n(A_n(\psi))\} < -\log \zeta$ ,
- (iii)  $\max \{\text{var}_n(A_n(\varphi_k)) : k \in \{1, \dots, K\}\} < \infty$ .

For each  $n \geq N$  we construct an associated iterated function system in the following way. Given  $\xi = \xi_1 \cdots \xi_n \in \mathcal{D}^n$  we let

$$S_\xi := S_{\xi_1} \circ \cdots \circ S_{\xi_n}.$$

It follows from the fact that  $(S_{ij})_{(i,j) \in \mathcal{D}}$  is an INC-system that  $(S_\eta)_{\eta \in \mathcal{D}^n}$  is also an INC-system. Moreover, it follows from conditions (i), (ii) and (iii) above that the potentials  $A_n(\varphi_1), \dots, A_n(\varphi_K)$  on  $(\mathcal{D}^n)^\mathbb{N} = \Sigma$ , together with the INC-system  $(S_\eta)_{\eta \in \mathcal{D}^n}$  satisfy the conditions of Lemma 4.1 with  $\sigma^n$  in place of  $\sigma$ ,  $A_n(\varphi_k)$  in place of  $\varphi_k$ ,  $S_n(\chi)$  in place of  $\chi$ ,  $S_n(\psi)$  in place of  $\psi$ ,  $\zeta^n$  in place of  $\zeta$  and  $\text{var}_n$  in place of  $\text{var}_1$ . We let,

$$\begin{aligned} E_{A_n(\varphi)}^{\sigma^n}(\alpha) &:= \left\{ \omega \in \Sigma : \lim_{l \rightarrow \infty} A_{ln}(\varphi_k)(\omega) = \alpha_k \text{ for all } k \leq K \right\} \\ J_{A_n(\varphi)}^{\sigma^n}(\alpha) &:= \Pi(E_{A_n(\varphi)}^{\sigma^n}(\alpha)). \end{aligned}$$

Note also that,

$$\begin{aligned} C_{\sigma^n}(S_n(\chi), S_n(\psi)) &:= \max \left\{ \frac{-\log \zeta^n}{-\log \zeta^n - \text{var}_n(S_n(\chi))}, \frac{-\log \zeta^n}{-\log \zeta^n - \text{var}_n(S_n(\psi))} \right\} \\ &= \max \left\{ \frac{-\log \zeta}{-\log \zeta - \text{var}_n(A_n(\chi))}, \frac{-\log \zeta}{-\log \zeta - \text{var}_n(A_n(\psi))} \right\} \\ &= C_\sigma^n(\chi, \psi). \end{aligned}$$

Thus, by Lemma 4.1 we have,  $\dim_{\mathcal{H}} J_{A_n(\varphi)}^{\sigma^n}(\alpha)$

$$\begin{aligned} &\leq C_{\sigma^n}(S_n(\chi), S_n(\psi)) \sup \left\{ D_{nq}(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^{nq}}^*(\Sigma), \left| \int A_{nq}(\varphi_k) d\mu - \alpha_k \right| < 3\text{var}_n(A_n(\varphi_k)), \right. \\ &\quad \left. \text{for } k \leq K \text{ and } \int A_{nq}(\varphi_k) d\mu > m \text{ for } K < k \leq M \right\} \\ &\leq C_{\sigma^n}^n(\chi, \psi) \sup \left\{ D_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \left| \int A_q(\varphi_k) d\mu - \alpha_k \right| < 3\text{var}_n(A_n(\varphi_k)), \right. \\ &\quad \left. \text{for } k \leq K \text{ and } \int A_{nq}(\varphi_k) d\mu > m \text{ for } K < k \leq M \right\}. \end{aligned}$$

Moreover, given  $\omega \in E_{\varphi}(\alpha)$  we have,  $\lim_{l \rightarrow \infty} A_l(\varphi_k)(\omega) = \alpha_k$  and hence  $\lim_{l \rightarrow \infty} A_{ln}(\varphi_k)(\omega) = \alpha_k$ . Thus,  $J_{\varphi}(\alpha) \subseteq J_{A_n(\varphi)}^{\sigma^n}(\alpha)$ . Hence,

$$\begin{aligned} \dim_{\mathcal{H}} J_{\varphi}(\alpha) &\leq C_{\sigma^n}^n(\chi, \psi) \sup \left\{ D_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \left| \int A_q(\varphi_k) d\mu - \alpha_k \right| < 3\text{var}_n(A_n(\varphi_k)), \right. \\ &\quad \left. \text{for } k \leq K \text{ and } \int A_{nq}(\varphi_k) d\mu > m \text{ for } K < k \leq M \right\}. \end{aligned}$$

□

We now require a lemma relating  $\sigma^q$ -invariant measures to  $\sigma$ -invariant measures.

**Lemma 4.3.** *Take  $\nu \in \mathcal{M}_{\sigma^q}^*(\Sigma)$  and let  $\mu = A_q(\nu)$ . Then,*

- (i)  $\mu \in \mathcal{M}_{\sigma}^*(\Sigma)$ ,
- (ii) If  $\nu \in \mathcal{E}_{\sigma^q}^*(\Sigma)$  then  $\mu \in \mathcal{E}_{\sigma}^*(\Sigma)$ ,
- (iii)  $h_{\mu}(\sigma) = q^{-1}h_{\nu}(\sigma^q)$ ,
- (iv)  $h_{\mu \circ \pi^{-1}}(\sigma_v) = q^{-1}h_{\nu \circ \pi^{-1}}(\sigma_v^q)$ ,
- (v)  $h_{\mu}(\sigma|\pi_v^{-1}\mathcal{A}_v) = q^{-1}h_{\nu}(\sigma^q|\pi^{-1}\mathcal{A})$ .

Moreover, given any  $\theta \in C(\Sigma)$ ,  $\theta^v \in C(\Sigma_v)$  we have,

- (vi)  $\int \theta d\mu = \int A_q(\theta) d\nu$ ,
- (vii)  $\int \theta^v d\mu \circ \pi^{-1} = \int A_q(\theta^v) d\nu \circ \pi^{-1}$ .

*Proof.* Parts (i), (ii), (iii) and (vi) follow from [JJOP, Lemma 2]. It is clear that  $\mu$  is compactly supported. Since  $\pi \circ \sigma = \sigma_v \circ \pi$  we have  $A_k(\nu \circ \pi^{-1}) = A_k(\nu) \circ \pi^{-1}$  and hence (iv) and (vii) also follow from [JJOP, Lemma 2]. Part (v) follows from parts (iii) and (iv) combined with the Abramov Rokhlin formula [AR]. □

The following proposition completes the proof of the upper bound.

**Proposition 4.1.** *Suppose we have countably many potentials  $(\varphi_k)_{k \in \mathbb{N}}$ . Then, for all  $\alpha = (\alpha_k)_{k \in \mathbb{N}} \in (\mathbb{R} \cup \{\infty\})^{\mathbb{N}}$  we have,*

$$\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq \lim_{m \rightarrow \infty} \sup \left\{ D(\mu) : \mu \in \mathcal{E}_{\sigma}^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}.$$

*Proof.* It suffices to show that for each  $m \in \mathbb{N}$  we have,

$$\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq \sup \left\{ D(\mu) : \mu \in \mathcal{E}_{\sigma}^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}.$$

Fix  $m \in \mathbb{N}$ . Without loss of generality we may assume that there are only  $m$  potentials  $\varphi_1, \dots, \varphi_m$ . If not, we consider the set,

$$(4.4) \quad E_{\varphi}^m(\alpha) := \left\{ \omega \in \Sigma : \lim_{n \rightarrow \infty} A_n(\varphi_k) = \alpha_k \text{ for } k \leq m \right\},$$

$$(4.5) \quad J_{\varphi}^m(\alpha) := \Pi(E_{\varphi}^m(\alpha)),$$

and note that  $E_{\varphi}(\alpha) \subseteq E_{\varphi}^m(\alpha)$  and hence  $\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq \dim_{\mathcal{H}} J_{\varphi}^m(\alpha)$ . Finally we may reorder our potentials so that there is some  $K \leq m$  such that for all  $k \leq K \leq M$  we have  $\alpha_k \in \mathbb{R}$  and for  $K < k \leq M$ ,  $\alpha_k = \infty$ . Now we are in precisely the position of Lemma 4.2, so

$$\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq C_{\sigma}^m(\chi, \psi) \sup \left\{ D_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{M}_{\sigma^q}^*(\Sigma), \left| \int A_q(\varphi_k) d\mu - \alpha_k \right| < 3\text{var}_n(A_n(\varphi_k)), \right. \\ \left. \text{for } k \leq K \text{ and } \int A_{nq}(\varphi_k) d\mu > m \text{ for } K < k \leq M \right\}.$$

Since  $\lim_{n \rightarrow \infty} \text{var}_n(A_n(\chi)) = \lim_{n \rightarrow \infty} \text{var}_n(A_n(\psi)) = \lim_{n \rightarrow \infty} \text{var}_n(A_n(\varphi_k)) = 0$  for all  $k \leq m$ , and hence  $\lim_{n \rightarrow \infty} C_{\sigma}^n(\chi, \psi) = 1$ , we have,

$$\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq \sup \left\{ D_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \left| \int A_q(\varphi_k) d\mu - \alpha_k \right| < \frac{1}{m}, \right. \\ \left. \text{for } k \leq K \text{ and } \int A_q(\varphi_k) d\mu > m \text{ for } K < k \leq M \right\}.$$

Recall that for  $\gamma \in \mathbb{R} \cup \{+\infty\}$  and  $l \in \mathbb{N}$  we let

$$(4.6) \quad B_l(\gamma) := \begin{cases} \{x : |x - \gamma| < \frac{1}{l}\} & \text{if } \gamma \in \mathbb{R} \\ (l, +\infty) & \text{if } \gamma = \infty. \end{cases}$$

So we may rewrite the above inequality as

$$\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq \sup \left\{ D_q(\mu) : q \in \mathbb{N}, \mu \in \mathcal{E}_{\sigma^q}^*(\Sigma), \int A_q(\varphi_k) d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}.$$

Finally, by Lemma 4.3, given  $\nu \in \mathcal{E}_{\sigma^q}^*(\Sigma)$  we may choose  $\mu \in \mathcal{E}_{\sigma}^*(\Sigma)$  with  $D(\mu) = D_q(\nu)$  and  $\int \varphi_k d\mu = \int A_q(\varphi_k) d\nu$ . Thus,

$$\dim_{\mathcal{H}} J_{\varphi}(\alpha) \leq \sup \left\{ D(\mu) : \mu \in \mathcal{E}_{\sigma}^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}.$$

This completes the proof of the upper bound.  $\square$

## 5. PRELIMINARY LEMMAS FOR THE LOWER BOUND

**5.1. Dimension Lemmas.** In this section we shall relate the symbolic local dimension of a measure to the local dimension of its projection. This will enable us to apply the following standard lemma.

**Lemma 5.1.** *Let  $\nu$  be a finite Borel measure on some metric space  $X$ . Suppose we have  $J \subseteq X$  with  $\nu(J) > 0$  such that for all  $x \in J$*

$$\liminf_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} \geq d.$$

*Then  $\dim_{\mathcal{H}} J \geq d$ .*

*Proof.* See [F2] Proposition 2.2. □

Given subsets  $A, B \subseteq \mathbb{R}$  by  $A \leq B$  we mean  $x \leq y$  for all  $x \in A$  and  $y \in B$ . We shall say that the digit set  $\mathcal{D}$  is wide if there exists  $(i_1, j_1), (i_2, j_2) \in \mathcal{D}$  with  $i_1 = i_2$  and  $j_1 \neq j_2$ . It follows that there exists an  $i' \in \mathcal{A}$  together with pairs  $j_-^1, j_-^2, j_0^1, j_0^2, j_+^1, j_+^2 \in \mathbb{N}$  so that

$$(5.1) \quad f_{i'j_-^1} \circ f_{i'j_-^2}([0, 1]) \leq f_{i'j_0^1} \circ f_{i'j_0^2}([0, 1]) \leq f_{i'j_+^1} \circ f_{i'j_+^2}([0, 1]).$$

Now define,

$$(5.2) \quad W_n(\omega) := \min \{l > n + 1 : \omega_l = (i', j_0^1), \omega_{l+1} = (i', j_0^2)\} - n,$$

and

$$(5.3) \quad R_n(\omega) := \max \left\{ -\log \inf_{x \in [0, 1]} |f'_{\omega_n}(x)| : \nu \leq n + W_n(\omega) + 1 \right\}.$$

**Lemma 5.2.** *Let  $\mu$  be a finite Borel measure on  $\Sigma$  supported on  $\pi^{-1}(\{\tau\})$  for some  $\tau \in \Sigma_v$ . Let  $\nu := \mu \circ \Pi^{-1}$  the corresponding projection on  $\Lambda$ . Suppose  $\mathcal{D}$  is wide. Then for all  $x = \Pi(\omega) \in \Lambda$  with  $\pi(\omega) = \tau$  and  $\lim_{n \rightarrow \infty} R_n(\omega)n^{-1} = \lim_{n \rightarrow \infty} R_n(\omega)W_n(\omega)n^{-1} = 0$ ,*

$$\liminf_{r \rightarrow 0} \frac{\log \nu(B(x, r))}{\log r} \geq \liminf_{n \rightarrow \infty} \frac{-\log \mu([\omega|n])}{S_n(\chi)(\omega)}.$$

*Proof.* Suppose that  $\mathcal{D}$  is wide and fix  $x = (x_1, x_2) = \Pi(\omega) \in \Lambda$  for some  $\omega = ((i_\nu, j_\nu))_{\nu \in \mathbb{N}} \in \pi^{-1}(\{\tau\})$  with  $\lim_{n \rightarrow \infty} R_n(\omega)n^{-1} = \lim_{n \rightarrow \infty} R_n(\omega)W_n(\omega)n^{-1} = 0$ . First note that since  $\mu$  is supported on  $\pi^{-1}(\{\tau\})$ ,  $\nu$  is supported on  $\mathbb{R} \times \{x_2\}$ . Define,

$$a_0 := \inf \{ |f'_d(z)| : z \in [0, 1], d \in \{(i', j_+^1), (i', j_+^2), (i', j_-^1), (i', j_-^2)\} \}.$$

Take  $n \in \mathbb{N}$ . By the definition of  $W_n(\omega)$  the finite string  $\omega_{n+W_n(\omega)} = (i', j_0^1)$  and  $\omega_{n+W_n(\omega)+1} = (i', j_0^2)$ . Now let,

$$\begin{aligned} \eta_0 &:= (\omega_{n+1}, \dots, \omega_{n+W_n(\omega)-1}, (i', j_0^1), (i', j_0^2)) \\ \eta_+ &:= (\omega_{n+1}, \dots, \omega_{n+W_n(\omega)-1}, (i', j_+^1), (i', j_+^2)) \\ \eta_- &:= (\omega_{n+1}, \dots, \omega_{n+W_n(\omega)-1}, (i', j_-^1), (i', j_-^2)). \end{aligned}$$

It follows from (5.1) that one of the following holds;

$$\begin{aligned} f_{\omega|n} \circ f_{\eta_-}([0, 1]) &\leq f_{\omega|n} \circ f_{\eta_0}([0, 1]) \leq f_{\omega|n} \circ f_{\eta_+}([0, 1]), \\ f_{\omega|n} \circ f_{\eta_+}([0, 1]) &\leq f_{\omega|n} \circ f_{\eta_0}([0, 1]) \leq f_{\omega|n} \circ f_{\eta_-}([0, 1]). \end{aligned}$$

Clearly each interval is contained within the interval  $f_{\omega|n}([0, 1])$ . Moreover, it follows from the definitions of  $W_n(\omega)$  and  $a_0$  that both  $f_{\eta_+}([0, 1])$  and  $f_{\omega|n} \circ f_{\eta_-}([0, 1])$  are of diameter at least  $\inf_{x \in [0, 1]} |f'_{\omega|n}(x)| e^{-W_n(\omega)R_n(\omega)} a_0^2$ . Thus, since  $x_1 \in f_{\omega|n} \circ f_{\eta_0}([0, 1]) = f_{\omega|n+W_n(\omega)+1}([0, 1])$  and  $x_2 \in g_{\mathbf{i}|n}([0, 1])$  we have,

$$(5.4) \quad B\left(x_1, \inf_{z \in [0, 1]} |f'_{\omega|n}(z)| e^{-W_n(\omega)R_n(\omega)} a_0^3\right) \subseteq f_{\omega|n}([0, 1]),$$

and hence,

$$(5.5) \quad \nu\left(B\left(x, \inf_{z \in [0, 1]} |f'_{\omega|n}(z)| e^{W_n(\omega)R_n(\omega)} a_0^3\right)\right) \leq \mu([\omega|n]),$$

since  $\nu$  is supported on  $\mathbb{R} \times \{x_2\}$ . So let

$$(5.6) \quad r_n := \inf_{z \in [0, 1]} |f'_{\omega|n}(z)| e^{W_n(\omega)R_n(\omega)} a_0^3.$$

Choose  $(z_n)_{n \in \mathbb{N}} \subset [0, 1]$  so that  $|f'_{\omega|n}(z_n)| = \inf_{z \in [0, 1]} |f'_{\omega|n}(z)|$ . Note that for all  $(\tau_1, \dots, \tau_n) \in \mathcal{D}^n$ ,  $\text{diam}(f_{\tau_1} \circ \dots \circ f_{\tau_n}([0, 1])) \leq \zeta^n$ , so since the family  $\{\log |f'_{ij}| : (i, j) \in \mathcal{D}\}$  is uniformly equicontinuous we have,

$$\begin{aligned} \frac{1}{n} \log \inf_{z \in [0, 1]} |f'_{\omega|n}(z)| &= \frac{1}{n} \log |f'_{\omega|n}(z_n)|, \\ &= \frac{1}{n} \sum_{l=1}^n \log |f'_{\omega_l}(f_{\omega_{l+1}} \circ \dots \circ f_{\omega_n}(z_n))|, \\ &= \frac{1}{n} \sum_{l=1}^n \left( \log |f'_{\omega_l}(\Pi(\sigma^l \omega))| + o(n-l) \right), \\ &= \frac{1}{n} \sum_{l=1}^n \log |f'_{\omega_l}(\Pi(\sigma^l \omega))| + o(n) \\ &= -\frac{1}{n} S_n(\chi)(\omega) + o(n). \end{aligned}$$

It follows that,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log r_n}{-S_n(\chi)(\omega)} &= \lim_{n \rightarrow \infty} \frac{\log \inf_{z \in [0, 1]} |f'_{\omega|n}(z)| e^{W_n(\omega)R_n(\omega)} a_0^3}{S_n(\chi)(\omega)} \\ &= \lim_{n \rightarrow \infty} \frac{-S_n(\chi)(\omega) + o(n) + W_n(\omega)R_n(\omega)}{-S_n(\chi)(\omega)} = 1. \end{aligned}$$

Therefore,

$$\liminf_{n \rightarrow \infty} \frac{\log \nu(B(x, r_n))}{\log r_n} \geq \liminf_{n \rightarrow \infty} \frac{\log \mu([\omega|n])}{-S_n(\chi)(\omega)}.$$

To conclude the proof of the lemma we observe that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{\log r_{n+1}}{\log r_n} &= \lim_{n \rightarrow \infty} \frac{S_{n+1}(\chi)(\omega)}{S_n(\chi)(\omega)}, \\
&= \lim_{n \rightarrow \infty} \frac{S_n(\chi)(\omega) + \log |f'_{\omega_{n+1}}(\Pi(\sigma^{n+1}\omega))|}{S_n(\chi)(\omega)}, \\
&= \lim_{n \rightarrow \infty} \frac{S_n(\chi)(\omega) + O(R_n(\omega))}{S_n(\chi)(\omega)} = 1.
\end{aligned}$$

□

We say that the digit set  $\mathcal{D}$  is tall if there exists  $(i_3, j_3), (i_4, j_4) \in \mathcal{D}$  with  $i_3 \neq i_4$ . It follows that there exists pairs  $i_-^1, i_-^2, i_0^1, i_0^2, i_+^1, i_+^2 \in \mathcal{A}$  so that, for all  $x, y, z \in [0, 1]$  we have,

$$(5.7) \quad g_{i_-^1} \circ g_{i_-^2}(x) \leq g_{i_0^1} \circ g_{i_0^2}(y) \leq g_{i_+^1} \circ g_{i_+^2}(z).$$

Define

$$(5.8) \quad T_n(\tau) := \min \{l > n : i_{l-1} = i_0^1, i_l = i_0^2\} - n.$$

**Lemma 5.3.** *Let  $\mu$  be a finite Borel measure on  $\Sigma_v$  and let  $\nu := \mu \circ \Pi_v^{-1}$  denote the corresponding projection. Suppose  $\mathcal{D}$  is tall. Then for all  $y = \Pi_v(\tau) \in \Pi_v(\Sigma_v)$  with  $\lim_{n \rightarrow \infty} T_n(\tau)n^{-1} = 0$ ,*

$$\liminf_{r \rightarrow 0} \frac{\log \nu(B(y, r))}{\log r} \geq \liminf_{n \rightarrow \infty} \frac{-\log \mu([\tau|n])}{S_n(\psi)(\tau)}.$$

*Proof.* Proceed as in Lemma 5.2 with  $b_{\min} := \max \{|| - \log g'_i ||_\infty : i \in \mathcal{A}\}$  in place of  $W_n(\omega)$ . □

## 6. CONVERGENCE LEMMAS

**Lemma 6.1.** *Given  $\mu \in \mathcal{M}_\sigma^*(\Sigma)$ ,  $\epsilon > 0$  and  $m \in \mathbb{N}$  we may obtain  $q \geq m$  and  $\nu \in \mathcal{B}_{\sigma^q}^\dagger(\Sigma)$  satisfying,*

- (i)  $\left| \frac{h_{\nu \circ \pi^{-1}}(\sigma_v^q)}{\int S_q(\psi) d\nu \circ \pi^{-1}} - \frac{h_{\mu \circ \pi^{-1}}(\sigma_v)}{\int \psi d\mu \circ \pi^{-1}} \right| < \epsilon,$
- (ii)  $\left| \frac{h_\nu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\chi) d\nu} - \frac{h_\mu(\sigma | \pi^{-1} \mathcal{A})}{\int \chi d\mu} \right| < \epsilon,$
- (iii)  $\left| \int A_q(\varphi_k) d\nu - \int \varphi_k d\mu \right| < \epsilon$  for all  $k \leq m,$
- (iv)  $\text{var}_n(A_n(\varphi_k)) < \frac{1}{m},$  for all  $n \geq q$  and all  $k \leq m,$
- (v)  $\max \{ \text{var}_n(A_n(\chi)), \text{var}_n(A_n(\psi)) \} < \frac{1}{m},$  for all  $n \geq q(m).$

*Proof.* First not that since  $\lim_{q \rightarrow \infty} \text{var}_q(A_q(\chi)) = \lim_{q \rightarrow \infty} \text{var}_q(A_q(\psi)) = 0$  and  $\lim_{q \rightarrow \infty} \text{var}_q(A_q(\varphi_k)) = 0$  for all  $k$  we may choose  $q_0 \geq m$  so that for  $n \geq q_0$ ,  $\max\{\text{var}_n(A_n(\chi)), \text{var}_n(A_n(\psi))\} < \frac{1}{m}$  and  $\text{var}_n(A_n(\varphi_k)) < \frac{1}{m}$  for  $k \leq m$ .

Fix  $\mu \in \mathcal{M}_\sigma^*(\Sigma)$ . Given  $q \in \mathbb{N}$  we let  $\tilde{\nu}_q \in \mathcal{B}_{\sigma^q}^*(\Sigma)$  denote the  $q$ -th level approximation of  $\nu$ . That is, given a cylinder  $[\omega_1 \cdots \omega_{nq}]$  of length  $nq$  we let

$$(6.1) \quad \tilde{\nu}_q([\omega_1 \cdots \omega_{nq}]) := \prod_{l=0}^{n-1} \nu([\omega_{lq+1} \cdots \omega_{(l+1)q}]).$$

By the Kolmogorov-Sinai theorem (see [W] Theorem 4.18) we then have,

$$(6.2) \quad \lim_{q \rightarrow \infty} q^{-1} h_{\tilde{\nu}_q}(\sigma^q) = h_\nu(\sigma),$$

$$(6.3) \quad \lim_{q \rightarrow \infty} q^{-1} h_{\tilde{\nu}_q \circ \pi^{-1}}(\sigma_v^q) = h_{\nu \circ \pi^{-1}}(\sigma_v).$$

Combining these two limits and applying the Abramov Rohklin formula [AR] gives,

$$(6.4) \quad \lim_{q \rightarrow \infty} q^{-1} h_{\tilde{\nu}_q}(\sigma^q | \pi^{-1} \mathcal{A}) = h_\nu(\sigma).$$

Since  $\mu$  and  $\tilde{\nu}_q$  agree on cylinders of length  $q$  we have  $|\int A_q(\varphi_k) d\tilde{\nu}_q - \int A_q(\varphi_k) d\mu| \leq \text{var}_q A_q(\varphi_k)$ , which tends to zero with  $q$  by the tempered distortion property. Moreover,  $\int A_q(\varphi_k) d\mu = \int \varphi_k d\mu$ , since  $\mu$  is  $\sigma$ -invariant. Hence,

$$(6.5) \quad \lim_{k \rightarrow \infty} \int A_q(\varphi_k) d\tilde{\nu}_q = \int \varphi d\nu,$$

for all  $k \leq m$ . The same argument also gives,

$$(6.6) \quad \lim_{k \rightarrow \infty} \int A_q(\chi) d\tilde{\nu}_q = \int \chi d\nu,$$

$$(6.7) \quad \lim_{k \rightarrow \infty} \int A_q(\psi) d\tilde{\nu}_q \circ \pi^{-1} = \int \psi d\nu \circ \pi^{-1}.$$

Consequently, by taking  $q \geq q_0$  sufficiently large, we may obtain  $q \in \mathbb{N}$  and  $\tilde{\nu} \in \mathcal{B}_{\sigma^q}^*(\Sigma)$  satisfying (i), (ii) and (iii) from the lemma. To obtain  $\nu \in \mathcal{B}_{\sigma^q}^*(\Sigma)$  satisfying (i), (ii) and (iii) we perturb  $\tilde{\nu}$  slightly to obtain  $\nu \in \mathcal{B}_{\sigma^q}^*(\Sigma)$  with  $\nu_q([\omega_1 \cdots \omega_q]) > 0$  for each  $(\omega_1, \dots, \omega_q) \in \mathcal{D}_0^q$  whilst using continuity to insure that (i), (ii) and (iii) still hold. Since  $q \geq q_0$ , (iv) and (v) also hold.  $\square$

Recall that we defined  $\mathcal{A}$  to be the Borel sigma algebra on  $\Sigma_v$ . Given any Borel probability measure  $\nu \in \mathcal{M}(\Sigma)$  and  $\omega \in \Sigma$  we let  $\nu_\omega^{\pi^{-1}\mathcal{A}}$  denote the conditional measure at  $\omega$  [EW, Section 5.3]. Since  $\pi^{-1}\mathcal{A}$  is countably generated there exists  $\Sigma' \subseteq \Sigma$  with  $\nu(\Sigma') = 1$  such that for all  $\omega^1, \omega^2 \in \Sigma'$  with  $\tau = \pi(\omega^1) = \pi(\omega^2)$  we have  $\nu_{\omega^1}^{\pi^{-1}\mathcal{A}} = \nu_{\omega^2}^{\pi^{-1}\mathcal{A}}$  and  $\nu_{\omega^1}^{\pi^{-1}\mathcal{A}}(\pi^{-1}\{\tau\}) = 1$

[EW, Theorem 5.14]. It follows that we can take a family of measures  $\{\nu_\tau\}_{\tau \in \Sigma_v} \subset \mathcal{M}(\Sigma)$  with  $\nu^\tau(\pi^{-1}\{\tau\}) = 1$  for all  $\tau \in \Sigma_v$  and

$$(6.8) \quad \nu = \int \nu^\tau d\nu \circ \pi^{-1}(\tau).$$

We shall make use of the following well known result which is essentially contained within [LY, Lemma 9.3.1]. We refer the reader to [Pa].

**Lemma 6.2** (Ergodic theorem of information theory). *Suppose  $T$  is an ergodic measure-preserving transformation of a Borel probability space  $(X, \mathcal{B}, m)$ . Let  $\xi$  be a countable partition with  $\bigvee_{l=0}^{\infty} T^{-l}\xi = \mathcal{B}$  and  $\mathcal{C} \subset \mathcal{B}$  a  $T$ -invariant sub sigma algebra. Then for  $m$  almost every  $x \in X$  we have,*

$$\lim_{n \rightarrow \infty} \frac{1}{n} I_m \left( \bigvee_{l=0}^{n-1} T^{-l}\xi \middle| \mathcal{C} \right) (x) \rightarrow h_m(T|\mathcal{C}).$$

*Proof.* See the proof of [LY, Lemma 9.3.1]. The Lemma is a mild generalisation of [Pa, Theorem 7, Chapter 2] and may be proven in the same way.  $\square$

Fix some finite subset  $\mathcal{D}_0 \subseteq \mathcal{D}$  such that,

- (i) If  $\mathcal{D}$  is tall then there exists some  $j^1, j^2 \in \mathbb{N}$  with  $(i_0^1, j^1), (i_0^2, j^1) \in \mathcal{D}_0$ , where  $i_0^1, i_0^2 \in \mathcal{A}$  are as in the definition of  $T_n(\omega)$ ,
- (ii) If  $\mathcal{D}$  is wide then  $(i', j_0^1), (i', j_0^2) \in \mathcal{D}_0$ , where  $(i', j_0^1), (i', j_0^2) \in \mathcal{D}$  are as in the definition of  $W_n(\omega)$ .

We shall let  $\mathcal{B}_{\sigma^q}^0(\Sigma)$  denote the set of  $\mu \in \mathcal{B}_{\sigma^q}^*(\Sigma)$  which satisfy,  $\mu([\omega_1, \dots, \omega_q]) > 0$  for all  $(\omega_1, \dots, \omega_q) \in \mathcal{D}_0^q$ .

**Lemma 6.3.** *Given  $\nu \in \mathcal{E}_{\sigma^q}^0(\Sigma)$ , the following convergences hold for  $\nu$  almost every  $\omega \in \Sigma$ ,*

- (i)  $\lim_{n \rightarrow \infty} A_{nq}(\psi)(\tau) = \int A_q(\psi) d\nu \circ \pi^{-1},$
- (ii)  $\lim_{n \rightarrow \infty} A_{nq}(\chi)(\omega) = \int A_q(\chi) d\nu,$
- (iii)  $\lim_{n \rightarrow \infty} A_{nq}(\varphi_k)(\omega) = \int A_q(\varphi_k) d\nu$  for all  $k \leq m,$
- (iv)  $\lim_{n \rightarrow \infty} -n^{-1} \log \nu \circ \pi^{-1}([\pi(\omega)_1 \dots \pi(\omega)_{nq}]) = h_{\nu \circ \pi^{-1}}(\sigma_v^q),$
- (v)  $\lim_{n \rightarrow \infty} -n^{-1} \log \nu^{\pi(\omega)}([\omega_1 \dots \omega_{nq}]) = h_\nu(\sigma^q | \pi^{-1}\mathcal{A}),$
- (vi)  $\lim_{n \rightarrow \infty} n^{-1} T_n(\tau) = 0,$  provided  $\mathcal{D}$  is tall,
- (vii)  $\lim_{n \rightarrow \infty} n^{-1} W_n(\omega) = 0,$  provided  $\mathcal{D}$  is wide.

*Proof.* Limits (i)-(iii) follow from Birkhoff's ergodic theorem. Indeed since  $\sigma^q$  is ergodic with respect to  $\nu$ ,  $\sigma_v^q$  is ergodic with respect to  $\nu \circ \pi^{-1}$ .



If we let  $\xi_v$  denote the partition of  $\Sigma_v$  into cylinder sets of length  $q$  and  $\mathcal{N} := \{\Sigma_v, \emptyset\}$  denote the null sigma algebra, then for each  $\tau \in \Sigma_v$  we have,

$$I_{\nu \circ \pi^{-1}} \left( \bigvee_{l=0}^{n-1} \sigma^{-lq} \xi_v \middle| \mathcal{N} \right) (\tau) = \log \nu \circ \pi^{-1}([\tau_1 \cdots \tau_{nq}]).$$

Thus (i) follows from Lemma 6.2. Similarly if we let  $\xi_h$  denote the partition of  $\Sigma$  into cylinder sets of length  $q$  then for each  $\omega \in \Sigma$  we have,

$$I_{\nu} \left( \bigvee_{l=0}^{n-1} \sigma^{-lq} \xi_h \middle| \pi^{-1} \mathcal{A} \right) (\omega) = \log \nu^{\pi(\omega)}([\omega_1 \cdots \omega_{nq}]),$$

so (ii) also follows from Lemma 6.2.

For each  $(\eta_1, \dots, \eta_q) \in \mathcal{D}_0^q$  we have,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \# \{l < n : (\omega_{lq+1}, \dots, \omega_{lq+q}) = (\eta_1, \dots, \eta_q)\} \\ = \nu([\eta_1, \dots, \eta_q]) > 0. \end{aligned}$$

Limits (vi) and (vii) follow.  $\square$

**Lemma 6.4.** *Given  $\nu \in \mathcal{E}_{\sigma^q}^0(\Sigma)$ , supported on some compact set  $K$ , along with constants  $\delta, \epsilon > 0$  and  $m \in \mathbb{N}$ , there exists  $N \in \mathbb{N}$  and  $U \subseteq \Sigma_v$  with  $\nu \circ \pi^{-1}(U) > 1 - \delta$  such that for all  $\tau \in U$  and all  $n \geq N$  we have,*

$$(i) \left| \frac{\log \nu \circ \pi^{-1}([\tau_1 \cdots \tau_{nq}])}{S_{nq}(\psi)(\tau)} + \frac{h_{\nu \circ \pi^{-1}}(\sigma_v^q)}{\int S_q(\psi) d\nu \circ \pi^{-1}} \right| < \epsilon,$$

$$(ii) T_n(\tau) < n\epsilon, \text{ provided } \mathcal{D} \text{ is tall.}$$

Moreover, for each  $\tau \in U$  there exists  $V_\tau \subseteq \pi^{-1}\{\tau\} \cap K$  with  $\nu^\tau(V_\tau) > 1 - \delta$  and for all  $\omega \in V_\tau$  and  $n \geq N$  we have,

$$(iii) \left| \frac{\log \nu^\tau([\omega_1 \cdots \omega_{nq}])}{S_{nq}(\chi)(\omega)} + \frac{h_\nu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\chi) d\nu} \right| < \epsilon,$$

$$(iv) W_n(\omega) < n\epsilon, \text{ provided } \mathcal{D} \text{ is wide,}$$

$$(v) \left| A_{nq}(\varphi_k)(\omega) - \int A_q(\varphi_k) d\nu \right| < \epsilon \text{ for all } k \leq m.$$

*Proof.* By Lemma 6.3 (i), (iv) and (vi) combined with Egorov's theorem, there exists a set  $U'' \subset \Sigma_v$  with  $\nu \circ \pi^{-1}(U'') > 1 - \delta/2$ , such that for all  $\tau \in U''$  and all  $n \geq N'$  we have,

$$(i) \left| \frac{\log \nu \circ \pi^{-1}([\tau_1 \cdots \tau_{nq}])}{S_{nq}(\psi)(\tau)} + \frac{h_{\nu \circ \pi^{-1}}(\sigma_v^q)}{\int S_q(\psi) d\nu \circ \pi^{-1}} \right| < \epsilon,$$

$$(ii) T_n(\tau) < n\epsilon, \text{ provided } \mathcal{D} \text{ is tall.}$$

By Lemma 6.3 (ii), (iii), (v) and (vii) we may take  $U' \subset U''$  with  $\nu \circ \pi^{-1}(U') = \nu \circ \pi^{-1}(U'')$  such for all  $\tau \in U'$ ,  $\nu^\tau$  is supported on  $\pi^{-1}\{\tau\} \cap K$  and for  $\nu^\tau$  almost all  $\omega \in \subseteq \pi^{-1}\{\tau\} \cap K$  we have,

$$(iii)', \lim_{n \rightarrow \infty} \frac{\log \nu^\tau([\omega_1 \cdots \omega_{nq}])}{S_{nq}(\chi)(\omega)} = \frac{h_\nu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\chi) d\nu},$$

$$(iv)', \lim_{n \rightarrow \infty} n^{-1} W_n(\omega) = 0, \text{ provided } \mathcal{D} \text{ is wide,}$$

$$(iv)', \lim_{n \rightarrow \infty} A_{nq}(\varphi_k)(\omega) = \int A_q(\varphi_k) d\nu \text{ for all } k \leq m.$$

Applying Egorov's theorem once more, we obtain for each  $\tau \in U'$  a set  $V_\tau \subseteq \pi^{-1}\{\tau\} \cap K$  with  $\nu^\tau(V_\tau) > 1 - \delta$  such that for all  $\omega \in V_\tau$  and  $n \geq N'(\tau)$  we have,

$$(iii) \left| \frac{\log \nu^\tau([\omega_1 \cdots \omega_{nq}])}{S_{nq}(\chi)(\omega)} + \frac{h_\nu(\sigma^q | \pi^{-1} \mathcal{A})}{\int S_q(\chi) d\nu} \right| < \epsilon,$$

$$(iv) W_n(\omega) < n\epsilon, \text{ provided } \mathcal{D} \text{ is wide,}$$

$$(iv) \left| A_{nq}(\varphi_k)(\omega) - \int A_q(\varphi_k) d\nu \right| < \epsilon \text{ for all } k \leq m.$$

Now choose  $U \subseteq U'$  with  $\nu \circ \pi^{-1}(U) > \nu \circ \pi^{-1}(U') - \delta/2 > 1 - \delta$  for which,

$$N := \max \{N(\tau) : \tau \in U\} < \infty.$$

□

## 7. PROOF OF THE LOWER BOUND

Throughout the proof of the lower bound we shall fix some  $\alpha = (\alpha_k)_{k \in \mathbb{N}} \subset \mathbb{R} \cup \{\infty\}$ . We define,

$$\delta(\alpha) := \lim_{m \rightarrow \infty} \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}.$$

In this section we shall prove the following.

**Proposition 7.1.**  $\dim_{\mathcal{H}} J_\varphi(\alpha) \geq \delta(\alpha)$ .

Clearly we may assume that  $\delta(\alpha) > -\infty$ . Thus, by a simple compactness argument there exists  $\delta_h(\alpha), \delta_v(\alpha) \in \mathbb{R}$  with  $\delta_h(\alpha) + \delta_v(\alpha) = \delta(\alpha)$ , along with a sequence of measures  $\{\mu_m\}_{m \in \mathbb{N}} \subset \mathcal{M}_\sigma^*(\Sigma)$  with

$$A(i) \frac{h_{\mu_m \circ \pi^{-1}}(\sigma_v)}{\int \psi d\mu_m \circ \pi^{-1}} > \delta_v(\alpha) - \frac{1}{3m},$$

$$\text{A(ii)} \quad \frac{h_{\mu_m}(\sigma|\pi^{-1}\mathcal{A})}{\int \chi d\mu_m} > \delta_h(\alpha) - \frac{1}{3m},$$

$$\text{A(iii)} \quad \int \varphi_k d\mu_m \in B_{3m}(\alpha_k) \text{ for } k \leq m.$$

Now choose  $\delta_m > 0$  for each  $m \in \mathbb{N}$  in such a way that  $\prod_{m=1}^{\infty} (1 - \delta_m) > 0$ .

By Lemma 6.1, for each  $m \in \mathbb{N}$ , there exists  $q(m) \geq m$  and  $\nu_m \in \mathcal{B}_{\sigma^{q(m)}}^0(\Sigma)$  satisfying,

$$\text{B(i)} \quad \frac{h_{\nu_m \circ \pi^{-1}}(\sigma_v^{q(m)})}{\int S_{q(m)}(\psi) d\nu_m \circ \pi^{-1}} > \delta_v(\alpha) - \frac{1}{2m},$$

$$\text{B(ii)} \quad \frac{h_{\nu_m}(\sigma^{q(m)}|\pi^{-1}\mathcal{A})}{\int S_{q(m)}(\chi) d\nu_m} > \delta_h(\alpha) - \frac{1}{2m},$$

$$\text{B(iii)} \quad \int A_{q(m)}(\varphi_k) d\nu_m \in B_{2m}(\alpha) \text{ for all } k \leq m,$$

$$\text{B(iv)} \quad \text{var}_n(A_n(\varphi_k)) < \frac{1}{m}, \text{ for all } n \geq q(m) \text{ and } k \leq m,$$

$$\text{B(v)} \quad \text{var}_n(A_n(\chi)), \text{var}_n(A_n(\psi)) < \frac{1}{m}, \text{ for all } n \geq q(m).$$

Since  $\nu_m \in \mathcal{B}_{\sigma^{q(m)}}^0(\Sigma)$  is compactly supported there is a finite digit set  $\mathcal{D}_m \subset \mathcal{D}$  such that  $\nu_m$  is supported on  $\mathcal{D}_m^{\mathbb{N}}$ . We define,

$$A(m) := \sup \left( \left\{ -\log \inf_{x \in [0,1]} |f_d| : d \in \mathcal{D}_m \right\} \cup \left\{ |\text{var}_1(\varphi_k)(\omega)| : \omega_1, \dots, \omega_{q(m)} \in \mathcal{D}_m \right\} \right).$$

Note that  $A(m)$  is finite since  $\mathcal{D}_m$  is finite and  $\text{var}_1(\varphi_k)$  is finite for all  $k \leq m$ .

For each  $m$  we define,  $\tilde{A}(m) := \prod_{l=1}^{m+1} A(l) + 1$ .

By Lemma 6.4, for each  $m \in \mathbb{N}$  we may take  $N(m) \in \mathbb{N}$  and  $U(m) \subseteq \Sigma_v$  with  $\nu_m \circ \pi^{-1}(U(m)) > 1 - \delta_m$  such that for all  $\tau \in U(m)$  and all  $n \geq N(m)$  we have,

$$\text{C(i)} \quad \frac{-\log \nu_m \circ \pi^{-1}([\tau_1 \cdots \tau_{nq}])}{S_{nq}(\psi)(\tau)} > \delta_v(\alpha) - \frac{1}{m},$$

$$\text{C(ii)} \quad \frac{T_n(\tau)}{n} < \frac{1}{m}, \text{ provided } \mathcal{D} \text{ is tall.}$$

Moreover, for each  $\tau \in U(m)$  there exists  $V_\tau(m) \subseteq \pi^{-1}\{\tau\} \cap \mathcal{D}_m^{\mathbb{N}}$  with  $\nu_m^\tau(V_\tau(m)) > 1 - \delta_m$  and for all  $\omega \in V_\tau(m)$  and  $n \geq N(m)$  we have,

$$\text{C(iii)} \quad \frac{-\log \nu_m^\tau([\omega_1 \cdots \omega_{nq}])}{S_{nq}(\chi)(\omega)} > \delta_h(\alpha) - \frac{1}{m},$$

$$\text{C(iv)} \quad \tilde{A}(m) \frac{W_n(\omega)}{n} < \frac{1}{m}, \text{ provided } \mathcal{D} \text{ is wide,}$$

$$\text{C(v)} \quad A_{nq}(\varphi_k)(\omega) \in B_m(\alpha_k) \text{ for all } k \leq m.$$

We now define a rapidly increasing sequence  $(\gamma_m)_{m \in \mathbb{N} \cup \{0\}}$  of natural numbers by  $\gamma_0 = 2N(1)$ ,  $\gamma_1 = 2N(2)$  and for  $m > 1$  we let

$$(7.1) \quad \gamma_m := (m+1)! \cdot \gamma_{m-1} \left( \prod_{l=1}^{m+1} N(l) \right) \left( \prod_{l=1}^{m+1} A(l) \right) \left( \prod_{l=1}^{m+1} q(l) \right) + \gamma_{m-1}.$$

We now define a measure  $\mathcal{W}$  on  $\Sigma_v$  by first defining  $\mathcal{W}$  on a semi-algebra of cylinders and then extending  $\mathcal{W}$  to a Borel probability measure on  $\Sigma_v$  via the Daniell-Kolmogorov consistency theorem ([W] Theorem 0.5). Given a cylinder  $[\tau_1 \cdots \tau_{\gamma_M}]$  of length  $\gamma_M$  for some  $M \in \mathbb{N}$  we define

$$(7.2) \quad \mathcal{W}([\tau_1 \cdots \tau_{\gamma_M}]) := \prod_{m=1}^M \nu_m \circ \pi^{-1}([\tau_{\gamma_{m-1}+1} \cdots \tau_{\gamma_m}]).$$

Define  $U \subseteq \Sigma_v$  by

$$(7.3) \quad U := \bigcap_{m=1}^{\infty} \left\{ \tau \in \Sigma_v : [\tau_{\gamma_{m-1}+1} \cdots \tau_{\gamma_m}] \cap U(m) \neq \emptyset \right\}.$$

For each  $\tau \in U$  and  $m \in \mathbb{N}$  we choose  $\hat{\tau}^m \in [\tau_{\gamma_{m-1}+1} \cdots \tau_{\gamma_m}] \cap U(m)$  and define a measure  $\mathcal{Z}^\tau$  on  $\Sigma$  by

$$(7.4) \quad \mathcal{Z}^\tau([\omega_1 \cdots \omega_{\gamma_M}]) := \prod_{m=1}^M \nu_m^{\hat{\tau}^m}([\omega_{\gamma_{m-1}+1} \cdots \omega_{\gamma_m}]).$$

**Lemma 7.1.** *For all  $\tau \in U$  we have  $\mathcal{Z}^\tau(\pi^{-1}\{\tau\}) = 1$ .*

*Proof.* For each  $m \in \mathbb{N}$  we have,

$$(7.5) \quad \nu_m^{\hat{\tau}^m}(\pi^{-1}[\tau_{\gamma_{m-1}+1} \cdots \tau_{\gamma_m}]) = \nu_m^{\hat{\tau}^m}(\pi^{-1}\{\hat{\tau}^m\}) = 1.$$

Hence, for each  $M \in \mathbb{N}$  we have

$$(7.6) \quad \mathcal{Z}^\tau(\pi^{-1}[\tau_1 \cdots \tau_{\gamma_M}]) = \prod_{m=1}^M \nu_m^{\hat{\tau}^m}(\pi^{-1}[\tau_{\gamma_{m-1}+1} \cdots \tau_{\gamma_m}]) = 1.$$

The lemma follows. □

For each  $\tau \in U$  we define  $V_\tau \subseteq \pi^{-1}\{\tau\}$  by

$$(7.7) \quad V_\tau := \bigcap_{m=1}^{\infty} \left\{ \omega \in \Sigma : [\omega_{\gamma_{m-1}+1} \cdots \omega_{\gamma_m}] \cap V_\tau(m) \neq \emptyset \right\}.$$

We also define,

$$(7.8) \quad S := \{\omega \in \Sigma : \pi(\omega) \in U \text{ and } \omega \in V_{\pi(\omega)}\}.$$

We shall show that  $S \subseteq E_\varphi(\alpha)$  and  $\dim_{\mathcal{H}} \Pi(S) \geq \delta(\alpha)$ .

**Lemma 7.2.**  $S \subseteq E_\varphi(\alpha)$ .

*Proof.* Note that it suffices to take  $\omega \in \Sigma$  with  $\pi(\omega) \in U$  and  $\omega \in V_{\pi(\omega)}$  and show that for each  $k \in \mathbb{N}$  we have  $\lim_{n \rightarrow \infty} A_n(\varphi_k)(\omega) = \alpha_k$ .

Given  $n \in \mathbb{N}$  we choose  $m(n) \in \mathbb{N}$  so that  $\gamma_{m(n)} \leq n < \gamma_{m(n)+1}$  and  $l(n) \in \mathbb{N}$  so that  $\gamma_{m(n)} + l(n)q(m(n)+1) \leq n < \gamma_{m(n)} + (l(n)+1)q(m(n)+1)$ . Since  $\lim_{n \rightarrow \infty} m(n) = \infty$  we may choose  $n(k) \in \mathbb{N}$  so that  $m(n) \geq k$  for all  $n \geq n(k)$ .

First lets suppose that  $\alpha_k \in \mathbb{R}$ . Given  $n \geq n(k)$ , either  $l(n) \leq N(m(n) + 1)$ , in which case

$$\begin{aligned} & |S_n(\varphi_k)(\omega) - n\alpha_k| \\ & \leq |S_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k)(\sigma^{\gamma_{m(n)-1}}\omega) - (\gamma_{m(n)} - \gamma_{m(n)-1})\alpha_k| \\ & \quad + \gamma_{m(n)-1}A(m(n)-1) + (N(m(n)+1)+1)q(m(n)+1)A(m(n)+1) \\ & \leq |S_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k)(\sigma^{\gamma_{m(n)-1}}\omega) - (\gamma_{m(n)} - \gamma_{m(n)-1})\alpha_k| + \frac{2\gamma_{m(n)}}{m(n)} \\ & \leq |S_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k)(\hat{\omega}^{m(n)}) - (\gamma_{m(n)} - \gamma_{m(n)-1})\alpha_k| \\ & \quad + (\gamma_{m(n)} - \gamma_{m(n)-1}) \text{var}_{\gamma_{m(n)} - \gamma_{m(n)-1}} \left( A_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k) \right) + \frac{2n}{m(n)} \\ & \leq \frac{2(\gamma_{m(n)} - \gamma_{m(n)-1})}{m(n)} + \frac{2n}{m(n)} \leq \frac{4n}{m(n)}. \end{aligned}$$

On the other hand, if  $l(n) > N(m(n) + 1)$  then we have,

$$|S_n(\varphi_k)(\omega) - n\alpha_k|$$

$$\begin{aligned}
&\leq |S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k)(\sigma^{\gamma_{m(n)-1}}\omega) - (\gamma_{m(n)} - \gamma_{m(n)-1})\alpha_k| \\
&\quad + |S_{l(n)q(m(n)+1)-\gamma_{m(n)}}(\varphi_k)(\sigma^{\gamma_{m(n)}}\omega) - (l(n)q(m(n)+1) - \gamma_{m(n)})\alpha_k| \\
&\quad + \gamma_{m(n)-1}A(m(n)-1) + q(m(n)+1)A(m(n)+1) \\
&\leq |S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k)(\sigma^{\gamma_{m(n)-1}}\omega) - (\gamma_{m(n)} - \gamma_{m(n)-1})\alpha_k| \\
&\quad + |S_{l(n)q(m(n)+1)-\gamma_{m(n)}}(\varphi_k)(\sigma^{\gamma_{m(n)}}\omega) - (l(n)q(m(n)+1) - \gamma_{m(n)})\alpha_k| + \frac{2\gamma_{m(n)}}{m(n)} \\
&\leq |S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k)(\hat{\omega}^{m(n)}) - (\gamma_{m(n)} - \gamma_{m(n)-1})\alpha_k| \\
&\quad + |S_{l(n)q(m(n)+1)-\gamma_{m(n)}}(\varphi_k)(\hat{\omega}^{m(n)+1}) - (l(n)q(m(n)+1) - \gamma_{m(n)})\alpha_k| \\
&\quad + (\gamma_{m(n)} - \gamma_{m(n)-1})\text{var}_{\gamma_{m(n)}-\gamma_{m(n)-1}}A_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k) \\
&\quad + (l(n)q(m(n)+1) - \gamma_{m(n)})\text{var}_{l(n)q(m(n)+1)-\gamma_{m(n)}}A_{l(n)q(m(n)+1)-\gamma_{m(n)}}(\varphi_k) + \frac{2\gamma_{m(n)}}{m(n)} \\
&\leq \frac{2(\gamma_{m(n)} - \gamma_{m(n)-1})}{m(n)} + \frac{2(l(n)q(m(n)+1) - \gamma_{m(n)})}{m(n)} + \frac{2\gamma_{m(n)}}{m(n)} \leq \frac{6n}{m(n)}.
\end{aligned}$$

Thus, for all  $n \geq n(k)$  we have,

$$|A_n(\varphi_k)(\omega) - \alpha_k| \leq \frac{6}{m(n)}.$$

Since  $\lim_{n \rightarrow \infty} m(n) = \infty$  the lemma holds when  $\alpha_k$  is finite.

Now suppose that  $\alpha_k = \infty$ . Given  $n \geq n(k)$ , either  $l(n) \leq N(m(n)+1)$ , in which case,

$$\begin{aligned}
S_n(\varphi_k)(\omega) &\geq S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k)(\sigma^{\gamma_{m(n)-1}}\omega) \\
&\geq S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k)(\omega^{\hat{m}(n)}) \\
&\quad - (\gamma_{m(n)} - \gamma_{m(n)-1}) \text{var}_{\gamma_{m(n)}-\gamma_{m(n)-1}} \left( A_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\varphi_k) \right) \\
&\geq (\gamma_{m(n)} - \gamma_{m(n)-1})m(n) - \frac{\gamma_{m(n)} - \gamma_{m(n)-1}}{m(n)} \\
&\geq nm(n) - (\gamma_{m(n)-1} + l(n)q(m(n)+1))m(n) - \frac{n}{m(n)} \\
&\geq nm(n) - \frac{2\gamma_{m(n)}}{m(n)} - \frac{n}{m(n)} \\
&\geq nm(n) - \frac{3n}{m(n)}.
\end{aligned}$$

On the other hand, if  $l(n) > N(m(n) + 1)$  then we have,

$$\begin{aligned}
S_n(\varphi_k)(\omega) &\geq S_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k)(\sigma^{\gamma_{m(n)-1}}\omega) + S_{l(n)q(m(n)+1) - \gamma_{m(n)}}(\varphi_k)(\sigma^{\gamma_{m(n)}}\omega) \\
&\geq S_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k)(\hat{\omega}^{m(n)}) + S_{l(n)q(m(n)+1) - \gamma_{m(n)}}(\varphi_k)(\hat{\omega}^{m(n)+1}) \\
&\quad - (\gamma_{m(n)} - \gamma_{m(n)-1}) \text{var}_{\gamma_{m(n)} - \gamma_{m(n)-1}} A_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\varphi_k) \\
&\quad - (l(n)q(m(n) + 1) - \gamma_{m(n)}) \text{var}_{l(n)q(m(n)+1) - \gamma_{m(n)}} A_{l(n)q(m(n)+1) - \gamma_{m(n)}}(\varphi_k) \\
&\geq (\gamma_{m(n)} - \gamma_{m(n)-1}) m(n) + (l(n)q(m(n) + 1) - \gamma_{m(n)}) m(n) - \frac{2n}{m(n)} \\
&\geq nm(n) - \frac{4n}{m(n)}.
\end{aligned}$$

Thus, for all  $n \geq n(k)$  we have,

$$A_n(\varphi_k)(\omega) \geq m(n) - \frac{4}{m(n)}.$$

Letting  $n \rightarrow \infty$  proves the lemma.  $\square$

**Lemma 7.3.**  $\mathcal{W}(U) > 0$  and for each  $\tau \in U$ ,  $\mathcal{Z}^\tau(V_\tau) > 0$ .

*Proof.*

$$\mathcal{W}(U) \geq \prod_{m=1}^{\infty} \nu_m \circ \pi^{-1}(U(m)) > \prod_{m=1}^{\infty} (1 - \delta_m) > 0.$$

Similarly for each  $\tau \in U$  we have,

$$\mathcal{Z}^\tau(V_\tau) \geq \prod_{m=1}^{\infty} \nu_m^{\hat{\tau}^m}(V_\tau(m)) > \prod_{m=1}^{\infty} (1 - \delta_m) > 0.$$

$\square$

**Lemma 7.4.** For all  $\tau \in U$  and all  $\omega \in V_\tau$  we have,

$$(i) \liminf_{n \rightarrow \infty} \frac{-\log \mathcal{W}([\tau_1 \cdots \tau_n])}{S_n(\psi)(\tau)} \geq \delta_v(\alpha),$$

$$(ii) \liminf_{n \rightarrow \infty} \frac{-\log \mathcal{Z}^\tau([\omega_1 \cdots \omega_n])}{S_n(\chi)(\omega)} \geq \delta_h(\alpha).$$

*Proof.* We prove (ii). The proof of (i) is similar. Take  $\tau \in U$  and  $\omega \in V_\tau$ . Given  $n \in \mathbb{N}$  we choose  $m(n) \in \mathbb{N}$  so that  $\gamma_{m(n)} \leq n < \gamma_{m(n)+1}$  and  $l(n) \in \mathbb{N}$  so that  $\gamma_{m(n)} + l(n)q(m(n) + 1) \leq n < \gamma_{m(n)} + (l(n) + 1)q(m(n) + 1)$ . If  $l(n) \leq N(m(n) + 1)$  then by C(iii) we have,

$$\begin{aligned}
-\log \mathcal{Z}^\tau([\omega_1 \cdots \omega_n]) &\geq -\log \nu_m^{\hat{\tau}^m}([\omega_{\gamma_{m(n)-1}+1} \cdots \omega_{\gamma_{m(n)}}]) \\
&\geq S_{\gamma_{m(n)} - \gamma_{m(n)-1}}(\chi)(\hat{\omega}^{m(n)}) \left( \delta_h(\alpha) - \frac{1}{m(n)} \right)
\end{aligned}$$

where

$$\hat{\omega}^{m(n)} \in [\omega_{\gamma_{m(n)-1}+1} \cdots \omega_{\gamma_{m(n)}}] \cap V_{\pi(\omega)}(m).$$

Moreover, using B(v) combined with  $\gamma_{m(n)} \leq n$  we have,

$$\begin{aligned} & S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\hat{\omega}^{m(n)}) \\ \geq & S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\sigma^{\gamma_{m(n)-1}}\omega) - (\gamma_{m(n)} - \gamma_{m(n)-1})\text{var}_{\gamma_{m(n)}-\gamma_{m(n)-1}}A_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi) \\ \geq & S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\sigma^{\gamma_{m(n)-1}}\omega) - \frac{\gamma_{m(n)} - \gamma_{m(n)-1}}{m(n)} \\ \geq & S_n(\chi)(\omega) - \gamma_{m(n)-1}A(m(n)-1) - (N(m(n)+1)+1)q(m(n)+1)A(m(n)+1) - \frac{n}{m(n)} \\ \geq & S_n(\chi)(\omega) - \frac{2\gamma_{m(n)}}{m(n)} - \frac{n}{m(n)} \\ \geq & S_n(\chi)(\omega) - \frac{3n}{m(n)}. \end{aligned}$$

On the other hand, if  $l(n) > N(m(n)+1)$  then by C(iii) we have

$$\begin{aligned} & -\log \mathcal{Z}^T([\omega_1 \cdots \omega_n]) \\ \geq & -\log \nu_m^{\hat{\tau}^m}([\omega_{\gamma_{m(n)-1}+1} \cdots \omega_{\gamma_{m(n)}}]) - \log \nu_m^{\hat{\tau}^m}([\omega_{\gamma_{m(n)}+1} \cdots \omega_{\gamma_{m(n)}+l(n)q(m(n)+1)}]), \\ \geq & (S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\hat{\omega}^{m(n)}) + S_{l(n)q(m(n)+1)}(\chi)(\hat{\omega}^{m(n)+1})) \left( \delta_h(\alpha) - \frac{1}{m(n)} \right), \end{aligned}$$

$$\begin{aligned} \hat{\omega}^{m(n)} & \in [\omega_{\gamma_{m(n)-1}+1} \cdots \omega_{\gamma_{m(n)}}] \cap V_{\pi(\omega)}(m(n)), \\ \hat{\omega}^{m(n)+1} & \in [\omega_{\gamma_{m(n)}+1} \cdots \omega_{\gamma_{m(n)+1}}] \cap V_{\pi(\omega)}(m(n)+1). \end{aligned}$$

As before, using B(v) combined with  $\gamma_{m(n)} + l(n)q(m(n)+1) \leq n$  we have,

$$\begin{aligned} S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\hat{\omega}^{m(n)}) & \geq S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\sigma^{\gamma_{m(n)-1}}\omega) - \frac{n}{m(n)} \\ S_{l(n)q(m(n)+1)}(\chi)(\hat{\omega}^{m(n)+1}) & \geq S_{l(n)q(m(n)+1)}(\chi)(\sigma^{\gamma_{m(n)}}\omega) - \frac{n}{m(n)}. \end{aligned}$$

It follows that,

$$\begin{aligned} & S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\hat{\omega}^{m(n)}) + S_{l(n)q(m(n)+1)}(\chi)(\hat{\omega}^{m(n)+1}) \\ \geq & S_{\gamma_{m(n)}-\gamma_{m(n)-1}}(\chi)(\sigma^{\gamma_{m(n)-1}}\omega) + S_{l(n)q(m(n)+1)}(\chi)(\sigma^{\gamma_{m(n)}}\omega) - \frac{2n}{m(n)} \\ \geq & S_n(\chi)(\omega) - \gamma_{m(n)-1}A(m(n)-1) - q(m(n)+1)A(m(n)+1) - \frac{2n}{m(n)} \\ \geq & S_n(\chi)(\omega) - \frac{4n}{m(n)}. \end{aligned}$$



Thus, for all  $n \in \mathbb{N}$  we have,

$$-\log \mathcal{Z}^\tau([\omega_1 \cdots \omega_n]) \geq \left( S_n(\chi)(\omega) - \frac{4n}{m(n)} \right) \left( \delta_h(\alpha) - \frac{1}{m(n)} \right).$$

Since  $\liminf_{n \rightarrow \infty} n^{-1} S_n(\chi)(\omega) \geq \xi > 0$  and  $\lim_{n \rightarrow \infty} m(n) = \infty$ , the lemma holds.  $\square$

**Lemma 7.5.** *For all  $\tau \in U$  and all  $\omega \in V_\tau$  we have,*

- (i)  $\lim_{n \rightarrow \infty} n^{-1} R_n(\omega) = 0$ , *provided  $\mathcal{D}$  is wide,*
- (ii)  $\lim_{n \rightarrow \infty} n^{-1} R_n(\omega) W_n(\omega) = 0$ , *provided  $\mathcal{D}$  is wide,*
- (iii)  $\lim_{n \rightarrow \infty} n^{-1} T_n(\tau) = 0$ , *provided  $\mathcal{D}$  is tall.*

*Proof.* We shall prove (i) and (ii) simultaneously. The proof of (iii) is similar to that of (ii).

Suppose that  $\mathcal{D}$  is wide and take  $\omega \in V_\tau$  with  $\tau \in U$ . Given  $n \in \mathbb{N}$  we choose  $m(n)$  to be the maximal natural number with  $\gamma_{m(n)-1} + N(m(n)) \leq n$ . Now suppose  $n(1 + (\tilde{A}(m(n))m(n))^{-1}) < \gamma_{m(n)}$ . Then we may choose  $\hat{\omega}^{m(n)} \in [\omega_{\gamma_{m(n)-1}+1} \cdots \omega_{\gamma_{m(n)}}] \cap V_\tau(m(n))$ . It follows from C(iv) that,

$$(7.9) \quad W_{n-\gamma_{m(n)-1}}(\hat{\omega}^{m(n)}) < \frac{n - \gamma_{m(n)-1}}{\tilde{A}(m(n))m(n)}$$

$$(7.10) \quad < \frac{n}{\tilde{A}(m(n))m(n)}$$

$$(7.11) \quad \leq \gamma_{m(n)} - n.$$

Thus, there is some  $l \in \{n+1, \dots, n(1 + (\tilde{A}(m(n))m(n))^{-1})\}$  with  $\omega_l = (i', j_0^1)$  and  $\omega_{l+1} = (i', j_0^2)$ . It follows that  $n + W_n(\omega) \leq \gamma_{m(n)}$  and hence,

$$R_n(\omega) \leq \max_{l \leq m(n)} A(l) \leq \tilde{A}(m(n)) < \frac{2\gamma_{m(n)-1}}{m(n)-1} < \frac{2n}{m(n)}.$$

Also,

$$R_n(\omega) W_n(\omega) \leq \tilde{A}(m(n)) \frac{n}{\tilde{A}(m(n))m(n)} < \frac{n}{m(n)}.$$

On the other hand, suppose that  $n(1 + (\tilde{A}(m(n))m(n))^{-1}) \geq \gamma_{m(n)}$ . Then we may choose  $\hat{\omega}^{m(n)+1} \in [\omega_{\gamma_{m(n)+1}} \cdots \omega_{\gamma_{m(n)+1}}] \cap V_\tau(m(n))$ . By C(iv) we have,

$$\begin{aligned} W_{N(m(n)+1)}(\hat{\omega}^{m(n)+1}) &< \frac{N(m(n)+1)}{\tilde{A}(m(n)+1)(m(n)+1)} \\ &\leq \gamma_{m(n)+1} - \gamma_{m(n)+1}. \end{aligned}$$

Thus, there is some  $l \in \left\{ n+1, \dots, \gamma_{m(n)} + N(m(n)+1) \left( \tilde{A}(m(n)+1)(m(n)+1) \right)^{-1} \right\}$ .

It follows that,

$$n + \mathcal{W}_n(\omega) \leq \gamma_{m(n)+1} + N(m(n)+1) \left( \tilde{A}(m(n)+1)(m(n)+1) \right)^{-1} < \gamma_{m(n)+2}.$$

Hence,

$$\begin{aligned} R_n(\omega) &\leq \max_{l \leq m(n)+1} A(l) \leq \tilde{A}(m(n)+1) \\ &< \frac{2\gamma_{m(n)}}{m(n)} < \frac{4n}{m(n)}. \end{aligned}$$

In addition, we have,

$$W_n(\omega) \leq \gamma_{m(n)} + \frac{N(m(n)+1)}{\tilde{A}(m(n)+1)(m(n)+1)} - n \leq \frac{n + N(m(n)+1)}{\tilde{A}(m(n))m(n)}.$$

Hence,

$$R_n(\omega)W_n(\omega) \leq \frac{n + N(m(n)+1)}{m(n)} \leq \frac{n + \gamma_{m(n)}}{m(n)} \leq \frac{3n}{m(n)}.$$

Thus, for all  $n \in \mathbb{N}$  we have,

$$\max \left\{ \frac{R_n(\omega)}{n}, \frac{R_n(\omega)W_n(\omega)}{n} \right\} \leq \frac{4}{m(n)}.$$

Letting  $n \rightarrow \infty$ , and hence  $m(n) \rightarrow \infty$ , proves the lemma.  $\square$

To complete the proof of the lower bound we require a version of Marstrand's slicing lemma.

**Lemma 7.6.** *Let  $J$  be any subset of  $\mathbb{R}^2$ , and let  $K$  be any subset of the  $y$ -axis. If  $\dim_{\mathcal{H}} J \cap (\mathbb{R} \times \{y\}) \geq t$  for all  $y \in K$ , then  $\dim_{\mathcal{H}} J \geq t + \dim_{\mathcal{H}} K$ .*

*Proof.* See [F3, Corollary 7.12].  $\square$

**Lemma 7.7.**  $\dim_{\mathcal{H}} \Pi(S) \geq \delta(\alpha)$ .

*Proof.* Recall that,  $S := \{\omega \in \Sigma : \pi(\omega) \in U \text{ and } \omega \in V_{\pi(\omega)}\}$ . It follows that,  $\Pi(S) = \bigcup_{\tau \in U} \Pi(V_{\tau})$ . Thus, for each  $y = \Pi_v(\tau) \in \Pi_v(U)$  with  $\tau \in U$  we have  $\Pi(S) \cap (\mathbb{R} \times \{y\}) = \Pi(V_{\tau})$ , since  $V_{\tau} \subseteq \pi^{-1}\{\tau\}$ . Hence, by Lemma 7.6 suffices to prove that  $\dim_{\mathcal{H}} \Pi_v(U) \geq \delta_v(\alpha)$  and for each  $\tau \in U$  we have  $\dim_{\mathcal{H}} \Pi(V_{\tau}) \geq \delta_h(\alpha)$ .

To see that  $\dim_{\mathcal{H}} \Pi_v(U) \geq \delta_v(\alpha)$  we consider two cases. Either  $\delta_v(\alpha) = 0$ , in which case the supposition is trivial since  $U \neq \emptyset$  by Lemma 7.3, or  $\delta_v(\alpha) > 0$ . It follows from A(i) that for some  $\mu \in \mathcal{M}_{\sigma}(\Sigma)$  we have  $h_{\mu \circ \pi^{-1}}(\sigma_v) > 0$ . Consequently  $\mathcal{D}$  must be tall. Thus, by Lemma 7.5 the hypotheses of Lemma 5.3 are satisfied, and so by Lemma 5.3 combined with Lemma 7.4 (i) for all  $y = \Pi_v(\tau) \in \Pi(U)$  we have,

$$\liminf_{r \rightarrow 0} \frac{\log \mathcal{W} \circ \Pi_v^{-1}(B(y, r))}{\log r} \geq \liminf_{n \rightarrow \infty} \frac{-\log \mathcal{W}([\tau|n])}{S_n(\psi)(\tau)} \geq \delta_v(\alpha).$$

Since, by Lemma 7.3,  $\mathcal{W} \circ \Pi_v^{-1}(\Pi_v(U)) \geq \mathcal{W}(U) > 0$ , by Lemma 5.1 we have  $\dim_{\mathcal{H}} \Pi_v(U) \geq \delta_v(\alpha)$ .

Now fix  $\tau \in U$ . To show that  $\dim_{\mathcal{H}} \Pi(V_\tau) \geq \delta_h(\alpha)$  we proceed similarly. If  $\delta_h(\alpha) = 0$  then by Lemma 7.3  $V_\tau \neq \emptyset$  and so the supposition is trivial. If on the other hand  $\delta_h(\alpha) > 0$  then by A(ii) we have  $h_\mu(\sigma|\pi^{-1}\mathcal{A}) > 0$  for some  $\mu \in \mathcal{M}_\sigma(\Sigma)$  and consequently  $\mathcal{D}$  must be wide. Thus, by Lemma 7.5 the hypotheses of Lemma 5.2 are satisfied, and so by Lemma 5.2 combined with Lemma 7.4 (ii) for all  $x = \Pi(\omega) \in \Pi(V_\tau)$  we have,

$$\liminf_{r \rightarrow 0} \frac{\log \mathcal{Z}^\tau \circ \Pi^{-1}(B(x, r))}{\log r} \geq \liminf_{n \rightarrow \infty} \frac{-\log \mathcal{Z}^\tau([\omega|n])}{S_n(\chi)(\omega)} \geq \delta_h(\alpha).$$

Again, by Lemma 7.3,  $\mathcal{Z}^\tau \circ \Pi^{-1}(\Pi(V_\tau)) \geq \mathcal{W}(V_\tau) > 0$ , by Lemma 5.1 we have  $\dim_{\mathcal{H}} \Pi(V_\tau) \geq \delta_h(\alpha)$ . Thus, by 7.6 the lemma holds.  $\square$

To complete the proof of the lower bound we note that by Lemma 7.2  $\Pi(S) \subseteq J_\varphi(\alpha)$ . Therefore, by Lemma 7.7 we have,

$$\dim_{\mathcal{H}} J_\varphi(\alpha) \geq \lim_{m \rightarrow \infty} \sup \left\{ D(\mu) : \mu \in \mathcal{M}_\sigma^*(\Sigma), \int \varphi_k d\mu \in B_m(\alpha_k) \text{ for } k \leq m \right\}.$$

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